

Mathematical Computations  
SPRING 2018, FACULTY OF MATHEMATICS NRU HSE  
**Exercises for 14.05.2018**

1. COMBINATORICS

*Problem 1.1.* Estimate the probability that a random permutation from  $S_{100}$  has no cycles of length greater than 50. (This is exactly the probability of success for the optimal strategy in the 100 prisoners problem [100].)

2. FUNCTIONS

*Problem 2.1.* Draw the filled Julia set of the map  $f : z \rightarrow z^2 + c$  for

(a)  $c = -1$ ,   (b)  $c = -0.12 \pm 0.74i$ ,   (c)  $c = i$ ,   (d)  $c = 0.36 - 0.37i$

The *filled Julia set* consists of all points  $z \in \mathbb{C}$  that do not escape to infinity under repeated iterations of  $f$ .

3. POLYNOMIALS

*Problem 3.1.* Check that the cyclotomic field  $\mathbb{Q}(e^{2\pi i/p})$  for a prime  $p$  contains either  $\sqrt{p}$  or  $\sqrt{-p}$ .

4. NUMBERS

*Problem 4.1.* Program a substitution cipher (for instance, the Caesar cipher). More precisely, the user enters a text message, and the program encodes the message by replacing every letter of the alphabet by another letter. The replacement rule is a given permutation of the alphabet (for instance, the permutation  $n \mapsto n + 3 \pmod{26}$  where letters are numbered by  $0, \dots, 25$ ).

*Problem 4.2.* Program the unbreakable one-time pad cipher. The user enters a text message, and the program translates this message into a sequence of numbers  $(n_1, \dots, n_l)$ , chooses a random key  $(k_1, \dots, k_l)$  of the same length (one-time pad), and encodes the message as  $((n_1 + k_1) \pmod{26}, \dots, (n_l + k_l) \pmod{26})$ .

*Problem 4.3.* The *Thue–Morse sequence* can be constructed by starting from 0 and using repeatedly the replacement rules  $0 \mapsto 01$  and  $1 \mapsto 10$ . Find the first 1024 terms of the Thue–Morse sequence. (This sequence provides the fairest way to share goods between two parties (in particular, the fairest penalty shoot-out). It can also be used to draw the Koch snowflake via Turtle graphics [Thue–Morse]).

REFERENCES

[100] *An Impossible Bet*, video by MinutePhysics, <https://youtu.be/eivG1BK1K6M>

[Thue–Morse] *The Fairest Sharing Sequence Ever*, video by Numberphile, <https://youtu.be/prh72BLNjIk>