

Mathematical Computations
SPRING 2018, FACULTY OF MATHEMATICS NRU HSE
Exercises for 14.05.2018

1. COMBINATORICS

Problem 1.1. Estimate the probability that a random permutation from S_{100} has no cycles of length greater than 50. (This is exactly the probability of success for the optimal strategy in the 100 prisoners problem [100].)

2. FUNCTIONS

Problem 2.1. Draw the filled Julia set of the map $f : z \rightarrow z^2 + c$ for

(a) $c = -1$, (b) $c = -0.12 \pm 0.74i$, (c) $c = i$, (d) $c = 0.36 - 0.37i$

The *filled Julia set* consists of all points $z \in \mathbb{C}$ that do not escape to infinity under repeated iterations of f .

3. POLYNOMIALS

Problem 3.1. Check that the cyclotomic field $\mathbb{Q}(e^{2\pi i/p})$ for a prime p contains either \sqrt{p} or $\sqrt{-p}$.

4. NUMBERS

Problem 4.1. Program a substitution cipher (for instance, the Caesar cipher). More precisely, the user enters a text message, and the program encodes the message by replacing every letter of the alphabet by another letter. The replacement rule is a given permutation of the alphabet (for instance, the permutation $n \mapsto n + 3 \pmod{26}$ where letters are numbered by $0, \dots, 25$).

Problem 4.2. Program the unbreakable one-time pad cipher. The user enters a text message, and the program translates this message into a sequence of numbers (n_1, \dots, n_l) , chooses a random key (k_1, \dots, k_l) of the same length (one-time pad), and encodes the message as $((n_1 + k_1) \pmod{26}, \dots, (n_l + k_l) \pmod{26})$.

Problem 4.3. The *Thue–Morse sequence* can be constructed by starting from 0 and using repeatedly the replacement rules $0 \mapsto 01$ and $1 \mapsto 10$. Find the first 1024 terms of the Thue–Morse sequence. (This sequence provides the fairest way to share goods between two parties (in particular, the fairest penalty shoot-out). It can also be used to draw the Koch snowflake via Turtle graphics [Thue–Morse]).

REFERENCES

[100] *An Impossible Bet*, video by MinutePhysics, <https://youtu.be/eivG1BK1K6M>

[Thue–Morse] *The Fairest Sharing Sequence Ever*, video by Numberphile, <https://youtu.be/prh72BLNjIk>