Mathematical Computations

Spring 2018, Faculty of Mathematics NRU HSE

Exercises for 28.04.2018

1. Functions

Problem 1.1. The Klein j-function (or j-invariant) is a unique holomorphic function on the upper half-plane characterized by the properties:

(1) it is invariant under the action of $PSL_2(\mathbb{Z})$ on the upper half-plane by fractional linear transformation,

(2) it has a simple zero at $e^{2\pi i/3}$,

(3) it takes value $12^3 = 1728$ at i.

Draw the level sets of $\mathrm{Im}(j)$ on the upper half-plane.

Problem 1.2. Find the first 5 terms of the Fourier series of the j-function, that is, expand $j(\tau)$ as a power series in $q = e^{2\pi i\tau}$. What is the constant coefficient? And the coefficient with q? (There is a surprising relation between the latter coefficient and the dimension of the smallest faithful representation of the Monster group [Monster]. This relation is just a small part of the *Monster moonshine* theory.)

2. Numbers

Problem 2.1. (a) Find the values of the j-function at $1+i$, $1+\sqrt{2}i$ and

$$\frac{1+i\sqrt{d}}{2}, \text{ where } d = 3, 7, 11, 19, 43, 67, 163$$

(these are the generators of the ring of integers for imaginary quadratic fields $\mathbb{Q}(\sqrt{-d})$ where d runs through all Heegner numbers).

(b) Check that all values in part (a) are perfect cubes.

(c) Combining parts (a) and (b) with the solution of Problem 1.2 explain why the number $e^{\pi\sqrt{163}}$ is almost integer (and $e^{\pi\sqrt{163}} - 744$ is almost a perfect cube).

Problem 2.2. (a) Express $64j(4i)$ as $a + b\sqrt{2}$ for integer a and b.

(b) Express $j(\sqrt{5}i)$ as $a + b\sqrt{5}$ for integer a and b.

3. Plane geometry

Problem 3.1. Program the manipulator that draws the barycenter and the medians for a given triangle. The user should be able to move all three vertices of a triangle inside a fixed region.

Problem 3.2. (a) Draw the Fano plane.

(b) Draw a realization of the finite projective plane over the field of 3 elements.

References

[Monster] *Monster group*, video by Numberphile, `https://youtu.be/jsSeoGpiWsw`