# PUSH-PULL OPERATORS ON CONVEX POLYTOPES

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Abstract. A classical result of Schubert calculus is an inductive description of Schubert cycles using divided difference (or push-pull) operators in Chow rings. We define convex geometric analogs of push-pull operators and describe their applications to the theory of Newton–Okounkov convex bodies. Convex geometric push-pull operators yield an inductive construction of Newton–Okounkov polytopes of Bott–Samelson varieties. In particular, we construct a Minkowski sum of Feigin–Fourier–Littelmann–Vinberg polytopes using convex geometric push-pull operators in type $A$.

## 1. Introduction

Let $X$ be a smooth algebraic variety, and $E \to X$ a vector bundle of rank two on $X$. Define the projective line fibration $Y = \mathbb{P}(E)$ as the variety of all lines in $E$. The natural projection $\pi : Y \to X$ induces the pull-back $\pi^* : A^*(X) \to A^*(Y)$ and the push-forward $\pi_* : A^*(Y) \to A^{*-1}(X)$ (aka *transfer* or *Gysin map*) in the (generalized) cohomology rings of $X$ and $Y$. The *push-pull* operator $\pi^*\pi_* : A^*(Y) \to A^{*-1}(Y)$ is a homomorphism of $A^*(X)$-modules, and can be described explicitly via Quillen–Vishik formula for any algebraic oriented cohomology theory $A^*$ (such as Chow ring, $K$-theory or algebraic cobordism). Push-pull operators are used extensively in representation theory (Demazure operators) and in Schubert calculus (divided difference operators). We discuss convex geometric counterparts of push-pull operators and their applications in the theory of Newton–Okounkov convex bodies and representation theory.

Convex geometric push-pull operators are motivated by the study of Newton–Okounkov polytopes of flag and Bott–Samelson varieties. Namely, if $Y = G/B$ is the complete flag variety for a connected reductive group $G$, then there is a natural projective line fibration $\pi_i : Y \to G/P_i$ for every simple root $\alpha_i$. Here $P_i \subset G$ denotes the minimal parabolic subgroup associated with $\alpha_i$, and $X = G/P_i$ the corresponding partial flag variety. For instance, if $G = GL_n(\mathbb{C})$ then points in $G/B$ can be identified with complete flags $(V^1 \subset V^2 \subset \ldots \subset V^{n-1} \subset \mathbb{C}^n)$ of subspaces, and the map $\pi_i$ forgets the subspace $V^i$. The corresponding push-pull operator $\partial_i : CH^*(Y) \to CH^{*-1}(Y)$ for Chow rings is often called *divided difference operator*, while the push-pull operator $D_i : K^*(Y) \to K^{*-1}(Y)$ for the $K$-theory is

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sometimes called *Demazure operator*. By $K^*(Y)$ we mean the Laurent polynomial ring $K_0(Y)[\beta,\beta^{-1}]$ graded by $\deg(\beta) = -1$ (the classical Demazure operator is obtained by specializing to $\beta = -1$, see [FK] for details).

A classical result of Schubert calculus [BGG, D] is an inductive description of Schubert cycles $[X_w] \in CH^*(Y)$ for all elements $w \in W$ in the Weyl group of $G$. Namely, if $w = s_{i_1}s_{i_2}\cdots s_{i_\ell}$ is a reduced decomposition of $w$ into the product of simple reflections, then

$$[X_w] = \partial_{i_\ell}\ldots\partial_{i_2}\partial_{i_1}[X_{\mathrm{id}}]. \tag{1}$$

A classical result in representation theory [A] is an inductive description of the Demazure character $\chi_w(\lambda)$ for every Schubert variety $X_w$ and a dominant weight $\lambda$ of $G$:

$$\chi_w(\lambda) = D_{i_1}D_{i_2}\ldots D_{i_\ell}(e^\lambda). \tag{2}$$

While formulas (1) and (2) look similar, there is no direct relation between them since in (1) operators are applied in the order opposite to that of (2).

In [Ki16], we defined convex geometric analogs of Demazure operators. They can be used to construct inductively polytopes $P_\lambda$ such that the sum of exponentials over lattice points in $P_\lambda$ yields the Demazure character $\chi_w(\lambda)$. Recently, Naoki Fujita showed that the Nakashima–Zelevinsky polyhedral realizations of crystal bases for a special reduced decomposition of the longest element $w_0 \in W$ can be constructed inductively using convex geometric Demazure operators in types $A_n$, $B_n$, $C_n$, $D_n$, and $G_2$ [Fu19]. In this setting, the convex geometric Demazure operators are applied in the same order as in (2).

In the present paper, we define different convex geometric analogs of push-pull operators that are more natural from the perspective of (1). The construction is motivated by the following question about Bott–Samelson varieties (cf. discussion at the end of [AB, Section 6]). Let $\underline{w_0} = s_{i_1}s_{i_2}\ldots s_{i_d}$ be a reduced decomposition of the longest element $w_0 \in W$ (here $d = \dim G/B$ ). Recall that the Bott–Samelson variety $R_{\underline{w_0}}$ (associated with the decomposition $\underline{w_0}$) together with a birational map $\pi_{\underline{w_0}} : R_{\underline{w_0}} \to G/B$, can be constructed inductively as a tower of projective line fibrations:

$$\{\mathrm{pt}\} = R_\varnothing \leftarrow R_{s_{i_1}} \leftarrow R_{s_{i_1}s_{i_2}} \leftarrow R_{s_{i_1}s_{i_2}s_{i_3}} \leftarrow \ldots \leftarrow R_{\underline{w_0}}.$$

How to define Newton–Okounkov polytopes $\Delta_v(R_{\underline{w_0}}, L)$ (or Newton polytopes of the corresponding toric degenerations) of the Bott–Samelson variety $R_{\underline{w_0}}$ so that these polytopes can also be constructed inductively using a simple inductive step? Furthermore, will this construction survive for semiample line bundles $L$ on $R_{\underline{w_0}}$ that actually come from $G/B$ (i.e., $L = \pi^*_{\underline{w_0}}L_\lambda$ for the line bundle $L_\lambda$ associated with a dominant weight $\lambda$ on $G/B$)?

More precisely, recall that to define the Newton–Okounkov convex bodies $\Delta_v(X, L)$ for line bundles $L$ on a variety $X$ we have to choose a valuation $v$ on the field of rational functions $\mathbb{C}(X)$. In particular, for $X = R_{\underline{w_0}}$ it is enough to choose a valuation on $\mathbb{C}(G/B)$. A family of Newton–Okounkov polytopes $\Delta_v(R_{\underline{w_0}}, L)$ parameterized by very ample line bundles (for a fixed valuation $v$) can be sometimes regarded as part of a continuous family of polytopes $P_t$ (that depend on a parameter $t$ inside a convex cone $C \subset \mathbb{R}^d$). So it is natural to ask whether polytopes $P_t$ degenerate to $\Delta_v(G/B, L_\lambda)$ as $t$ tends to a boundary point of $C$ (such a degeneration for $G = GL_3(\mathbb{C})$ is described in Example 3.3).

The first answer that comes to mind is to consider a tower of Grossberg–Karshon (GK) cubes [GK], which are exhibited as Newton–Okounkov polytopes $\Delta_v(R_{\underline{w_0}}, L)$ for "sufficiently ample" line bundles on $R_{\underline{w_0}}$ in [Fu18, HY16, HY18]. Indeed, GK cubes can be constructed inductively using the simplest version of convex geometric push-pull operators (see Section 3.2). Since the original construction of GK cubes corresponds to the order in Demazure character formula (2) and not to the order in formula (1) (that is, an induction argument uses terminal subwords of $\underline{w_0}$ and not the initial subwords), an alternative construction via push-pull operators yields a convex geometric relation between formulas (1) and (2) (see Section 5.1 for more details). In [GK], a toric degeneration of $R_{\underline{w_0}}$ was constructed as a Bott tower in the setting of complex analytic geometry (the resulting Newton polytopes are GK cubes). In [P, Section 1], this construction is rewritten in the algebraic setting. However, GK cubes do not degenerate to Newton–Okounkov polytopes of flag varieties. As was observed in [AB], the relationship between $G/B$ and $R_{\underline{w_0}}$ gets broken in the limit.

If we consider a more general version of convex geometric push-pull operators (see Section 3 for a complete definition) then it seems possible to produce inductively families of polytopes whose volume polynomials coincide with the degree polynomials of Bott–Samelson varieties for all semiample line bundles (see Section 5.2 for more details). In type $A$, we conjecture that the Newton–Okounkov polytopes of the Bott–Samelson variety $R_{\underline{w}}$ for all initial subwords $\underline{w}$ of the decomposition $\underline{w_0} = s_1(s_2s_1)\cdots(s_ns_{n-1}\cdots s_1)$ (for the valuation considered in [Ki18]) as well as the Minkowski sums of Feigin–Fourier–Littelmann–Vinberg (FFLV) polytopes can be constructed inductively using push-pull operators (this conjecture is verified in Section 5.2 for $\underline{w} = s_1s_2s_1s_3s_2$). Note that Newton–Okounkov polytopes of flag varieties occur naturally as degenerations of Minkowski sums of FFLV polytopes when all summands but one tend to zero.

Our main tool is the Khovanskii–Pukhlikov ring that can be associated with every convex polytope (see Section 2 for a reminder). We prove an analog of the projective bundle formula for the Khovanskii–Pukhlikov rings of push-pull polytopes (Section 4, Theorem 4.1). This theorem allows us to compare the volumes of polytopes with the degrees of Bott–Samelson varieties without actually computing any of them.

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## 2. Reminder on convex polytopes and Khovanskii–Pukhlikov rings

In this section, we remind the definition of the polytope ring associated with a convex polytope $P \subset \mathbb{R}^n$. This ring was originally introduced by Khovanskii and Pukhlikov in [KhP92] to give a convenient functorial description of the cohomology (or Chow) rings of smooth toric varieties. Later, Kaveh used Khovanskii–Pukhlikov ring to give a partial description of cohomology rings of spherical varieties [Ka11]. Recently, Khovanskii–Pukhlikov rings were applied to cohomology rings of toric bundles [HKhM20] and Schubert calculus on polyhedral realizations of Demazure crystals [Fu20]. A version of the Khovanskii–Pukhlikov ring (as an algebra over $\mathbb{R}$) can be defined for any convex polytope. Below we only consider the case of lattice polytopes and define the Khovanskii–Pukhlikov ring as a $\mathbb{Z}$-module.

Let $\mathbb{Z}^n \subset \mathbb{R}^n$ be the integer lattice. A convex lattice polytope $P \subset \mathbb{R}^n$ is a convex hull of finitely many points from $\mathbb{Z}^n$. Recall that two convex polytopes $P$ and $P'$ are called *analogous* if they have the same normal fan. In particular, there exist linear functions $h_1,\ldots, h_d$ on $\mathbb{R}^n$ such that any polytope $P'$ analogous to $P$ is given by inequalities:

$$h_i(x) \le H_i(P'),\ i = 1, \ldots, d$$

for some constants $H_1(P'),\ldots, H_d(P') \in \mathbb{R}$ that depend on $P'$. The collection of numbers $(H_1(P'), \ldots, H_d(P'))$ (called support numbers of $P'$) defines uniquely the polytope $P'$. If a polytope $P'$ is analogous to $P$ then there is a natural bijection between faces of $P'$ and faces of $P$. In the sequel, we denote by $F(P')$ the face of $P'$ that corresponds to a face $F \subset P$ (in particular, $F = F(P)$).

Denote by $S_P$ the set of all lattice polytopes analogous to $P$. This set can be endowed with the structure of a commutative semigroup using *Minkowski sum*

$$P_1 + P_2 = \{x_1 + x_2 \in \mathbb{R}^n \mid x_1 \in P_1,\ x_2 \in P_2\}.$$

We can embed the semigroup of convex polytopes into its Grothendieck group $L_P$, which is a lattice in $\mathbb{R}^d$. The elements of $L_P$ are called *virtual polytopes* analogous to $P$. They can be represented by $d$-tuples $(H_1, \ldots, H_d)$ such that $H_i = H_i(P_1)-H_i(P_2)$ where $P_1$ and $P_2$ are analogous to $P$. In general, the rank of $L_P$ is smaller than or equal to $d$. The equality holds if and only if $P$ is simple, that is, all facets of $P$ can be translated independently without changing the combinatorial type of $P$.

We now assume that $\dim P = n$, that is, $P$ is not contained in any hyperplane of $\mathbb{R}^n$. There is a homogeneous polynomial $vol_P$ of degree $n$ on the lattice $L_P$, called the *volume polynomial*. It is uniquely characterized by the property that its value $vol_P(P')$ on any convex polytope $P' \in S_P$ is equal to the volume of $P'$. We normalize the volume form on $\mathbb{R}^n$ so that the covolume of the lattice $\mathbb{Z}^n$ is equal to 1.

The symmetric algebra $\mathrm{Sym}(L_P)$ of $L_P$ can be thought of as the ring of differential operators with constant integer coefficients acting on $\mathbb{R}[L_P]$, the space of all polynomials on $L_P$. Let $(x_1, \ldots, x_\ell)$ (where $\ell \le d$) be coordinates on $L_P$. Denote $\frac{\partial}{\partial x_i}$ by $\partial_{x_i}$. In particular, if $P$ is simple, then $(H_1, \ldots, H_d)$ are coordinates on $L_P$, and there is a natural bijection between facets $\{h_i(x) = H_i(P)\}$ of $P$ and differential operators $\partial_i$ for $i = 1,\dots, d$.

*Definition* 1. The Khovanskii–Pukhlikov ring $R_P$ associated with the polytope $P$ is the quotient ring

$$\mathbb{Z}[\partial_1, \dots, \partial_\ell]/\mathrm{Ann}(vol_P).$$

Here $\mathrm{Ann}(vol_P)$ denotes the ideal in $\mathbb{Z}[\partial_1, \dots, \partial_\ell]$ that consists of all differential operators $D$ such that $D(vol_P) = 0$.

Since $\mathrm{Sym}(L_P)$ is a graded algebra, and the ideal $\mathrm{Ann}(vol_P) \subset \mathbb{Z}[\partial_1, \dots, \partial_\ell] \subset \mathrm{Sym}(L_P)$ is homogeneous, the ring $R_P$ is a graded $\mathbb{Z}$-module. It is of finite rank since all homogeneous polynomials in $\partial_1,\dots, \partial_\ell$ of degree greater than $n$ do annihilate $vol_P$ by degree reasons.

*Remark* 2.1. In what follows, we use that every (virtual) polytope $P' \in L_P$ defines a homogeneous element $\partial_{P'}$ of degree one in $R_P$. Namely, let $\partial_{P'}$ be the differential operator that takes a polynomial function $f$ on $L_P \otimes_{\mathbb{Z}} \mathbb{R}$ to its directional derivative with respect to $P'$, that is,

$$[\partial_{P'} f](Q) := \lim_{s\to 0} \frac{f(Q + P's)}{s}$$

for any $Q \in L_P$. Using vector calculus we can write an explicit formula for $\partial_{P'}$ in coordinates $(x_1, \dots, x_\ell)$ on $L_P$. If $(x_1(P'), \dots, x_\ell(P'))$ are coordinates of $P'$, which is regarded as an element of $L_P$, then $\partial_{P'} = x_1(P')\partial_1 + \dots + x_\ell(P')\partial_\ell$.

Note that if $P_1$ and $P_2$ are two polytopes in $L_P$ that can be obtained from each other by a parallel translation, then $\partial_{P_1} = \partial_{P_2}$ in $R_P$. Indeed, parallel translations do not change volumes of polytopes, so $(\partial_{P_1} - \partial_{P_2})$ considered as an element of $\mathrm{Sym}(L_P)$ does annihilate $vol_P$.

## 3. Definition of push-pull polytopes

In this section, we first define the notion of a *codimension two truncation* of a convex polytope, and then use this notion to construct push-pull polytopes. In short, the *push-pull polytope* $\Delta(P, Q_{\mathcal{F}})$ is the Cayley sum of two polytopes $P$ and $Q_{\mathcal{F}}$, where $P$ is analogous to a given polytope $\widehat{P}$, and $Q_{\mathcal{F}}$ is analogous to a codimension two truncation of $\widehat{P}$.

Let $\widehat{P} \subset \mathbb{R}^n$ be a convex polytope of dimension $n$. Let $\mathcal{F} = \{F_1, \dots, F_k\}$ be a (possibly empty) collection of codimension two faces of $\widehat{P}$. For every $i = 1,\dots, k$, choose a linear function $\psi_i(x)$ on $\mathbb{R}^n$ such that $\psi_i$ takes a constant value $\Psi_i(\widehat{P})$ on $F_i$, and $\psi_i(x) \le \Psi_i(\widehat{P})$ for all $x \in \widehat{P}$. Fix a constant $\Psi_i < \Psi_i(\widehat{P})$. A *codimension two truncation* $Q_{\mathcal{F}}$ of $\widehat{P}$ is obtained from $\widehat{P}$ by intersecting $\widehat{P}$ with half-spaces as follows:

$$Q_{\mathcal{F}} = \widehat{P} \cap \left( \bigcup_{i=1}^{k} \{\psi_i(x) \le \Psi_i\} \right).$$

In particular, $Q_{\mathcal{F}}$ has $k$ extra facets $\Gamma_i = \widehat{P}\cap\{\psi_i(x) = \Psi_i\}$ for $i = 1,\dots, k$ (in addition to facets that come from $\widehat{P}$). We always assume that $\Psi_i$ is sufficiently close to $\Psi_i(\widehat{P})$ so that if we replace every $\Psi_i$ by $s\Psi_i+(1-s)\Psi_i(\widehat{P})$ then the combinatorial type of $Q_{\mathcal{F}}$ does not change for all $s\in(0,1]$. In general, the combinatorial type of $Q_{\mathcal{F}}$ depends significantly on the choice of functions $\psi_i(x)$ and constants $\Psi_i$. In applications, we will usually make specific choices.

*Definition* 2. Let $P$ be a convex polytope analogous to $\widehat{P}$. The *push-pull polytope* $\Delta(P,Q_{\mathcal{F}})\subset\mathbb{R}^{n+1} = \mathbb{R}^n\times\mathbb{R}$ is the *Cayley sum* of polytopes $P$ and $Q_{\mathcal{F}}$. More precisely, $\Delta(P,Q_{\mathcal{F}})$ is the convex hull of the following set:

$$(P\times 1)\cup(Q_{\mathcal{F}}\times 0).$$

This definition is slightly different from the usual definition of the Cayley sum of polytopes. The reason for this modification will be seen from Examples 3.2 and 3.3 below.

In what follows, we mostly consider not the particular polytope $\Delta(P,Q_{\mathcal{F}})$ but the whole family of polytopes analogous to $\Delta(P,Q_{\mathcal{F}})$. It is easy to check that if $P_1$ is analogous to $P$ (and hence to $\widehat{P}$), then $\Delta(P,Q_{\mathcal{F}})$ is analogous to $\Delta(P+P_1,Q_{\mathcal{F}}+P_1)$. In particular, the space of all virtual polytopes analogous to $\Delta(P,Q_{\mathcal{F}})$ depends only on the difference $Q_{\mathcal{F}}-P$. Here and later we use minus sign to denote the subtraction in the Grothendieck group of convex polytopes with respect to Minkowski sum (and not the set-theoretic difference).

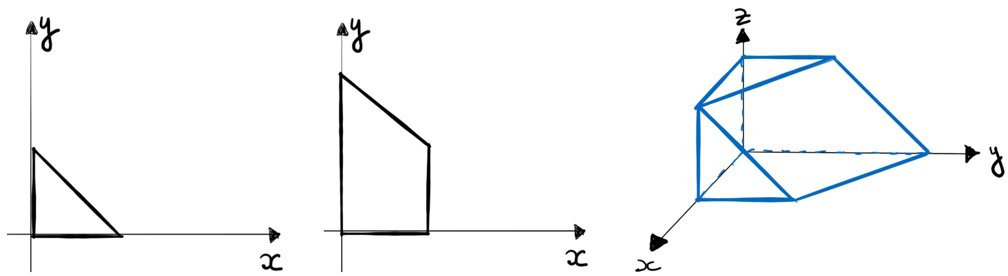
Below we consider motivating examples of push-pull polytopes.

3.1. **Minkowski sum with a segment.** A special codimension two truncation of $P$ can be obtained using the Minkowski sum with a segment. Let $I\subset\mathbb{R}^n$ be a segment, and $P'$ a polytope analogous to $P$. It is easy to show that the Minkowski sum $Q_{\mathcal{F}} = P'+I$ is a codimension two truncation of a polytope $\widehat{P}$ analogous to $P$. Indeed, the facets of $P'+I$ are either parallel to facets of $P'$ or equal to Minkowski sums $F+I$ where $F\subset P'$ is a face of codimension two. Then $\Delta(P,Q_{\mathcal{F}})$ is a convex hull of

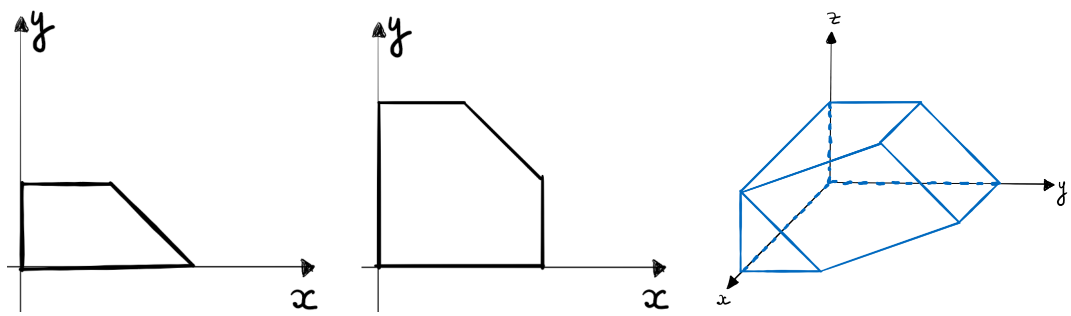
$$(P\times 1)\cup((P'+I)\times 0).$$

*Example* 3.1. In the simplest case $n = 1$ and $I = [0,1]$, the polytopes $P$ and $P+I\subset\mathbb{R}$ are also segments. If $P = [a,b]$, then $Q_{\mathcal{F}} = [a,b+1]$ and $\Delta(P,Q_{\mathcal{F}})$ is a trapezoid with the vertices $(a,0)$, $(b+1,0)$, $(a,1)$, $(b,1)$.

*Example* 3.2. Let $n = 2$, and $P\subset\mathbb{R}^2$ the triangle with the vertices $(0,0)$, $(1,0)$, $(0,1)$. Take $I = [(0,0),(0,1)]$, and $P' = P$. Then $Q_{\mathcal{F}} = P+I$ is analogous to the trapezoid obtained from $\widehat{P} = 2P$ by truncating the vertex $(2,0)$ (=codimension two face). The push-pull polytope $\Delta(P,Q_{\mathcal{F}})\subset\mathbb{R}^3$ is the FFLV polytope in type $A_2$ corresponding to the weight $\rho$ (see [FFL] for the definition of FFLV polytopes in type $A$ and their representation-theoretic meaning). The figure below shows $P$, $Q_{\mathcal{F}}$ and $\Delta(P,Q_{\mathcal{F}})$, respectively (from left to right).




*Example* 3.3. cf. [An13, Section 6.4] Let $n = 2$, and $P \subset \mathbb{R}^2$ the trapezoid with the vertices $(0,0)$, $(2,0)$, $(0,1)$, $(1,1)$. Take $I = [(0,0),(0,1)]$, and $P' = P$. Then $Q_{\mathcal{F}} = P + I$ is analogous to a 5-gon obtained from $\widehat{P}$ by truncating the vertex $(3,0)$. Here $\widehat{P}$ is the trapezoid with the vertices $(0,0)$, $(3,0)$, $(0,2)$, $(1,2)$. The push-pull polytope $\Delta(P, Q_{\mathcal{F}}) \subset \mathbb{R}^3$ is the Minkowski sum of the FFLV polytope from Example 3.2 and the segment $J = [(0,0,0),(1,0,0)]$. By shrinking $J$ we get a degeneration to the FFLV polytope. The figure below shows $P$, $Q_{\mathcal{F}}$ and $\Delta(P, Q_{\mathcal{F}})$, respectively (from left to right).




Example 3.3 produces the Newton–Okounkov polytope of a Bott–Samelson variety in type $A_2$ which does degenerate to the Newton–Okounkov polytope of the flag variety $GL_3/B$ (see [An13, Section 6.4] or [Ki18, Example 3.2]). An example in type $A_3$ will be considered in Section 5.2.

3.2. **Analogous $P$ and $Q_{\mathcal{F}}$.** In what follows, we use a well-known correspondence between projective toric varieties and their Newton (or moment) polytopes. With a full-dimensional lattice polytope $P \subset \mathbb{R}^n$ we associate a projective toric variety $X_P$ of dimension $n$ as in [CLS, Definition 2.3.14]. Suppose that $\mathcal{F} = \varnothing$, that is, no codimension two faces are truncated, and $Q_{\mathcal{F}} = \widehat{P}$ is analogous to $P$. Below we exhibit the push-pull polytope $\Delta(P, Q_{\mathcal{F}})$ as the Newton polytope of a projective line fibration $Y = \mathbb{P}(E)$ over $X_P$ where $E = L_1 \oplus L_2$ is a sum of two torus-invariant line bundles on $X_P$ (in particular, $Y$ is also a toric variety). For instance, Example 3.1 corresponds to the case where $X_P = \mathbb{P}^1$ is a projective line and $Y = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$ is a Hirzebruch surface.

Recall that the Picard group of $X_P$ can be identified with the Grothendieck group of lattice polytopes analogous to $P$ modulo parallel translations (see [CLS, Section 6.2] for details). In particular, when $X_P$ is smooth, this correspondence is used to define the isomorphism $CH^*(X_P) \simeq R_P$. Namely, if $L(P')$ is the line bundle corresponding to the virtual polytope $P'$ then the first Chern class $c_1(L(P'))$ gets mapped to $\partial_{P'}$ (see Remark 2.1 for the definition of $\partial_{P'}$). Let $L_1$ and $L_2$ be the line bundles corresponding to the polytopes $P$ and $Q_{\mathcal{F}}$, respectively. Then Newton polytopes of very ample line bundles on $Y$ are analogous to $\Delta(P, Q_{\mathcal{F}})$. This toric interpretation of $\Delta(P, Q_{\mathcal{F}})$ follows easily from an explicit description of Newton polytopes of projective embeddings of $Y$ in [G, Example 6.3]. Note that since $\mathbb{P}(E) = \mathbb{P}(E \otimes M)$ for any line bundle $M$ on $X$, the variety $Y$ depends only on the line bundle $L := L_1^* \otimes L_2$, which corresponds to the (possibly virtual) polytope $Q_{\mathcal{F}} - P$.

In particular, the construction of GK cubes [GK] motivated by Demazure character formula (2) for Bott–Samelson varieties can also be reproduced in the spirit of (1) using convex-geometric push-pull operators. We discuss this approach in more detail in Sections 5.1 and 5.3.

*Remark* 3.4. Suppose that $E$ is a non-split rank two vector bundle on a toric variety $X$. It will be interesting to check whether a Newton–Okounkov convex polytope of $Y = \mathbb{P}(E)$ can be obtained using the general version of push-pull operators. For instance, since the flag variety $GL_3(\mathbb{C})/B$ can also be regarded as $\mathbb{P}(T\mathbb{P}^2) =: Y$ where $T\mathbb{P}^2 =: E$ is the tangent bundle of the projective plane $\mathbb{CP}^2 =: X$ we can obtain the Newton–Okounkov polytope of $Y$ using Example 3.2 (cf. [G, Example 6.1]).

3.3. **Volume polynomial of $Q_{\mathcal{F}}$.** In what follows, we will need the following partial description of the volume polynomial of $Q_{\mathcal{F}}$. Let $s$ be a positive real number such that $s \leq 1$. Define the (possibly nonconvex and disconnected) polytope $\overline{Q_{\mathcal{F}}}(s) \subset \mathbb{R}^n$ as the union of convex polytopes

$$\bigcup_{i=1}^{k} \{\psi_i(x) \geq s\Psi_i + (1-s)\Psi_i(\widehat{P})\} \cap \widehat{P}.$$

Informally speaking, if we are cutting slices from $\widehat{P}$ continuously until we get $Q_{\mathcal{F}}$, then $Q_{\mathcal{F}}(s)$ will consist of those points of $\widehat{P} \setminus Q_{\mathcal{F}}$ that are already chopped at the moment $s$. In particular, $\mathrm{vol}(\widehat{P}) = \mathrm{vol}(Q_{\mathcal{F}}) + \mathrm{vol}(\overline{Q_{\mathcal{F}}}(1))$, as $\widehat{P} = Q_{\mathcal{F}} \cup \overline{Q_{\mathcal{F}}}(1)$, and $Q_{\mathcal{F}} \cap \overline{Q_{\mathcal{F}}}(1) = \bigcup_{i=1}^{k} \Gamma_i$. In general,

$$\mathrm{vol}(\overline{Q_{\mathcal{F}}}(s)) = \left(\sum_{i=1}^{k} a_i \mathrm{vol}_{n-2}(F_i(\widehat{P}))\right) \frac{s^2}{2} + \sum_{j=3}^{n} g_j(\widehat{P}) s^j.$$

where $a_1, \ldots, a_k \in \mathbb{R}$ and $g_j(\widehat{P})$ is a linear combination of volumes of certain codimension $j$ faces of $\widehat{P}$. This implies the following formula for the volume of the

Minkowski sum $sQ_{\mathcal{F}} + (1-s)\widehat{P}$:

$$\mathrm{vol}(sQ_{\mathcal{F}} + (1-s)\widehat{P}) = \mathrm{vol}(\widehat{P}) - \left(\sum_{i=1}^{k} a_i \mathrm{vol}_{n-2}(F_i(\widehat{P}))\right)\frac{s^2}{2} - \sum_{j=3}^{n} g_j(\widehat{P})s^j. \quad (3)$$

Here we use that $sQ_{\mathcal{F}} + (1-s)\widehat{P} = \widehat{P} \setminus \overline{Q_{\mathcal{F}}}(s)$. In particular, if $s = 1$ we get the formula for $\mathrm{vol}(Q_{\mathcal{F}})$.

## 4. Khovanskii–Pukhlikov ring of a push-pull polytope

In this section, we show that the Khovanskii–Pukhlikov rings of $P$ and of $\Delta := \Delta(P, Q_{\mathcal{F}})$ are related in the same manner as cohomology rings $H^*(X)$ and $H^*(Y)$ of smooth varieties $X$ and $Y$, whenever $Y$ is a projectivization of a rank two vector bundle over $X$. We use notation of Sections 2 and 3.

Let $R_P$ and $R_\Delta$ denote the Khovanskii–Pukhlikov rings of polytopes $P$ and $\Delta$, respectively. More precisely, suppose that $R_P$ is defined using the lattice $L_P$, and $L_P$ is spanned by polytopes $P_1,\ldots, P_\ell$. Define $L_\Delta$ as the lattice spanned by $\Delta(P + P_i, Q_{\mathcal{F}} + P_i)$ for $i = 1,\ldots, \ell$, and by an extra generator $C$. The virtual polytope $C$ can be informally thought of as the cone over $Q_{\mathcal{F}} - P$. To give a formal definition we will need the following description of polytopes analogous to $\Delta$.

*Definition* 3. Let $P'$ be a polytope analogous to $P$, and $s$ a positive real number. Suppose that $P' - sP = u_1P_1 + \ldots + u_\ell P_\ell$, where all $u_i$ are nonnegative (this condition ensures that $P' + s(Q_{\mathcal{F}} - P)$ is a true polytope). Define the polytope $\Delta(s, P') \subset \mathbb{R}^n \times \mathbb{R}$ as the convex hull of

$$(P' \times s) \cup ((P' + s(Q_{\mathcal{F}} - P)) \times 0).$$

In particular, $\Delta(1, P) = \Delta$. It is easy to check that the polytopes $\Delta(s, P')$ and $\Delta$ are analogous, and that the following identity holds:

$$\Delta(s, \sum_{i=1}^{\ell} s_iP_i) + \Delta(t, \sum_{i=1}^{\ell} t_iP_i) = \Delta(s + t, \sum_{i=1}^{\ell}(s_i + t_i)P_i).$$

Let $L_\Delta$ be the Grothendieck group of the semigroup of polytopes $\Delta(s, P')$. Then the lattice $L_\Delta$ has a unique basis $C$, $\Delta_1,\ldots, \Delta_\ell$, such that the polytope $\Delta(s, P')$ in this basis has coordinates $(s, s_1, \ldots, s_\ell)$. In particular, $C$ can be defined as the formal difference of polytopes $\Delta(2, P)$ and $\Delta$.

Consider the monomorphism of lattices $L_P \hookrightarrow L_\Delta$, which sends $P_i$ to $\Delta_i$ for $i = 1,\ldots, \ell$. In what follows, we identify $L_P$ with the sublattice of $L_\Delta$ using this monomorphism. Define the polynomial function $\mathrm{vol}_{\mathcal{F}}$ on $L_P$ by the condition

$$\mathrm{vol}_{\mathcal{F}}(P') = \sum_{i=1}^{k} a_i \mathrm{vol}_{n-2}(F_i(P'))$$

for all $P' \in L_P$ that are analogous to $P$. The coefficients $a_i$ here are the same as in the description of $\mathrm{vol}(\overline{Q_{\mathcal{F}}}(s))$ in Section 3.3. Similarly, we define polynomial functions $g_j$ on $L_P$ for $j = 3,\ldots, n$ so that formula (3) from Section 3.3 holds for all $P' \in L_P$. Define the polynomial $q(s, P')$ on $\mathbb{R} \times L_P$ by the formula:

$$q(s, P') = \mathrm{vol}_{\mathcal{F}}(P')\frac{s^2}{2} + \sum_{j=3}^{n} g_j(P')s^j.$$

**Theorem 4.1.** *Suppose that there exist a homogeneous element $D_{\mathcal{F}} \in R_P$ of degree two and homogeneous elements $D_j \in R_P$ of degrees $j = 3,\ldots, n$ such that $D_{\mathcal{F}}(\mathrm{vol}_P) = \mathrm{vol}_{\mathcal{F}}$ and $D_j(\mathrm{vol}_P) = g_j$. Put*

$$c_1 = \partial_{\widehat{P}} - \partial_P, \quad c_2 = D_{\mathcal{F}}.$$

*If $q(s, P')$ satisfies the condition:*

$$\left(\frac{\partial^2}{\partial s^2} - c_1\frac{\partial}{\partial s} + c_2\right) q(s, P' + s(\widehat{P} - P)) = \mathrm{vol}_{\mathcal{F}}(P' + s(\widehat{P} - P)), \tag{4}$$

*then there is a ring isomorphism*

$$R_P[x]/(x^2 - c_1x + c_2) \to R_\Delta; \quad x \mapsto \frac{\partial}{\partial s}.$$

*In particular, $c_1$ and $c_2$ can be thought of as analogs of the Chern classes of a rank two vector bundle.*

*Remark* 4.2. Note that if $P$ is simple, then there always exists a homogeneous element $D_{\mathcal{F}} \in R_P$ of degree two such that $D_{\mathcal{F}}(\mathrm{vol}_P) = \mathrm{vol}_{\mathcal{F}}$. Indeed, every face $F \subset P$ of codimension two is the intersection of two facets $\Gamma_1$ and $\Gamma_2$. Since $P$ is simple, every facet $\Gamma \subset P$ defines a homogeneous element $\partial_\Gamma \in R_P$ of degree one. Put $D = \partial_{\Gamma_1}\partial_{\Gamma_2}$. It is easy to check that $D(\mathrm{vol}_P) = a\mathrm{vol}_F$ for some constant $a \in \mathbb{R}$ that depends on the polytope $P$. Similarly, one can show that $D_3,\ldots, D_n$ also exist for simple $P$.

However, if $P$ is not simple then the existence of $D_{\mathcal{F}}$ and $D_3,\ldots, D_n$ can not be taken for granted.

*Proof.* First, we construct a ring monomorphism $\varphi : R_P \hookrightarrow R_\Delta$. Put $\partial_{s_i} = \frac{\partial}{\partial s_i}$ for $i = 1,\ldots, \ell$, and $\partial_s = \frac{\partial}{\partial s}$. Let $D$ be a polynomial in $\partial_{s_1},\ldots, \partial_{s_\ell}$. Since $L_P \subset L_\Delta$, the differential operator $D$ can also be regarded as the polynomial $\varphi(D)$ in $\partial_s$, $\partial_{s_1},\ldots,$ $\partial_{s_\ell}$. It remains to check that the map $\varphi$ is well-defined, that is, $D(\mathrm{vol}_P) = 0$ implies $\varphi(D)(\mathrm{vol}_\Delta) = 0$. By definition of $\Delta(s, P')$ and formula (3) for the volume of $Q_{\mathcal{F}}$ from Section 3, we have the following explicit formula for the volume polynomial $\mathrm{vol}_\Delta$:

$$\mathrm{vol}_\Delta(\Delta(s, P')) = \int_0^s \mathrm{vol}_P(x(t)) - q(t, x(t))dt,$$

where $x(t) = P' + t(\widehat{P} - P)$. Since $D(\mathrm{vol}_P) = 0$ and $\mathrm{vol}_{\mathcal{F}} = D_{\mathcal{F}}(\mathrm{vol}_P)$ we have that $D(\mathrm{vol}_{\mathcal{F}}) = D_{\mathcal{F}}D(\mathrm{vol}_P) = 0$. Similarly, we have that $D(g_i) = D_iD(\mathrm{vol}_P) = 0$. Hence, $\varphi(D)(\mathrm{vol}_\Delta) = 0$.

Second, we show that

$$(\partial_s - \partial_{\widehat{P}-P})\partial_s \mathrm{vol}_\Delta = -D_\mathcal{F}\mathrm{vol}_\Delta.$$

Using the explicit formula for $\mathrm{vol}_\Delta(\Delta(s,P'))$ we get

$$(\partial_s - \partial_{\widehat{P}-P})\partial_s \mathrm{vol}_\Delta = (\partial_s - \partial_{\widehat{P}-P})\left(\mathrm{vol}_P(x(s)) - q(s,x(s))\right).$$

By definition of the directional derivative $\partial_{\widehat{P}-P}$ (see Remark 2.1) we have that $(\partial_s - \partial_{\widehat{P}-P})\mathrm{vol}_P(x(s)) = 0$. Hence,

$$(\partial_s - \partial_{\widehat{P}-P})\left(\mathrm{vol}_P(x(s)) - q(s,x(s))\right) = -(\partial_s - \partial_{\widehat{P}-P})q(s,x(s)).$$

By condition (4) the right hand side is equal to $-D_\mathcal{F}\mathrm{vol}_\Delta(\Delta(s,P'))$. Indeed,

$$D_\mathcal{F}\mathrm{vol}_\Delta(\Delta(s,P')) = \int_0^s D_\mathcal{F}\mathrm{vol}_P(x(t)) - D_\mathcal{F}q(t,x(t))dt =$$

$$= \int_0^s \mathrm{vol}_\mathcal{F}(x(t)) - D_\mathcal{F}q(t,x(t))dt = \int_0^s \partial_t(\partial_t - \partial_{\widehat{P}-P})q(t,x(t))dt.$$

Finally, we establish the isomorphism $\Phi : R_P[x]/(x^2 - c_1x + c_2) \simeq R_\Delta$ by sending $x$ to $\partial_s$. This yields a well-defined ring homomorphism

$$\Phi : g + hx \mapsto \varphi(g) + \varphi(h)\partial_s,$$

since we already checked that $\partial_s^2 - c_1\partial_s + c_2 = 0$ in $R_\Delta$. Clearly, $\Phi$ is surjective since $R_\Delta$ is generated by $\partial_s$ and $\varphi(R_P)$. It is injective since $\partial_s \notin \varphi(R_P)$. □

*Remark* 4.3. Condition (4) is void if $\mathcal{F} = \varnothing$. Also, it holds trivially in the case where $g_3 = \ldots = g_n = 0$ (that is, $q(s,P')$ is a quadratic polynomial in $s$), and $c_1c_2 = c_2^2 = 0$ in $R_P$. This is the case in Examples 3.2 and 3.3 for dimension reasons. An example when condition (4) is nontrivial will be given in Section 5.2 (see Remark 5.3).

From a geometric viewpoint, condition (4) is "local", that is, depends only on the structure of the polytope $P$ in a sufficiently small neighborhood of the union of faces $\bigcup_{F_i\in\mathcal{F}} F_i$. Because of this, it is sometimes easier to verify (4) than more "global" statements about $P$ and the volume polynomial $\mathrm{vol}_P$.

As a byproduct of the proof of Theorem 4.1 we get the following

**Corollary 4.4.** *The volume polynomial* $\mathrm{vol}_\Delta$ *of the push-pull polytope satisfies the differential equation:*

$$F'' - (\partial_{\widehat{P}} - \partial_P)F' + D_\mathcal{F}F = 0,$$

*where an unknown function $F$ is a function of two variables $s \in \mathbb{R}$ and $x \in L_P \otimes \mathbb{R} = \mathbb{R}^\ell$, and $F'$ is the derivative of $F$ with respect to $s$.*

*Example* 4.5. We continue Example 3.3. A polytope $P'$ is analogous to $P$ if $P'$ is given by inequalities

$$0 \le (x - x_0); \quad 0 \le (y - y_0) \le b; \quad (x - x_0) + (y - y_0) \le a + b.$$

Choose a basis $P_1$, $P_2$, $P_3$, $P_4$ in $L_P$ so that $P'$ has coordinates $(a, b, x_0, y_0)$ in this basis. In particular, $\partial_{s_1} = \frac{\partial}{\partial a} := \partial_a$ and $\partial_{s_2} = \frac{\partial}{\partial b} := \partial_b$. Note that virtual polytopes $P_1$ and $P_2$ can also be interpreted as true polytopes in this case, namely, $P_1=\{0\le x\le 1;\ y=0\}$ is a segment, and $P_2=\{0\le x,y,x+y\le 1\}$ is a triangle. Indeed, $P'$ is a parallel translation of the Minkowski sum $aP_1+bP_2$. Since the area of $P'$ does not change under parallel translations we have that $\partial_{s_3}\mathrm{vol}_P=\partial_{s_4}\mathrm{vol}_P=0$. In other words, the volume polynomial $\mathrm{vol}_P=ab+\frac{b^2}{2}$ does not depend on $x_0$ and $y_0$. It is easy to check that the differential operators $(\partial_b^2-\partial_a\partial_b)$ and $\partial_a^2$ annihilate $\mathrm{vol}_P$. Hence, the Khovanskii–Pukhlikov ring $R_P$ is isomorphic to the quotient of the polynomial ring $\mathbb{Z}[\partial_a,\partial_b]$ by the ideal $(\partial_a^2,\partial_b^2-\partial_a\partial_b)$.

We now compute $c_1$ and $c_2$. We have that $\widehat{P}-P=P_2$, hence, $\partial_{\widehat{P}-P}=\partial_b$. Since $\mathcal{F}$ consists of a single vertex of $\widehat{P}$, we may choose any $D_{\mathcal{F}}$ such that $D_{\mathcal{F}}\mathrm{vol}_P=1$. For instance, take $D_{\mathcal{F}}=\partial_a\partial_b$. Hence, the statement of Theorem 4.1 reduces to the following isomorphisms:

$$R_\Delta\simeq R_P[x]/(x^2-\partial_b x+\partial_a\partial_b)\simeq\mathbb{Z}[\partial_s,\partial_a,\partial_b]/(\partial_a^2,\partial_b^2-\partial_a\partial_b,\partial_s^2-\partial_b\partial_s+\partial_a\partial_b),$$

which can be easily checked by direct calculation. Indeed, the volume polynomial $\mathrm{vol}_\Delta$ is equal to $s(ab+\frac{b^2}{2})+\frac{s^2}{2}(a+b)$. Clearly, the differential operator $\partial_s^2-\partial_b\partial_s+\partial_a\partial_b$ annihilates $\mathrm{vol}_\Delta$.

## 5. Applications to Schubert calculus and representation theory

5.1. **Formula for the Schubert cycles and Demazure character formula.** We now explain a simple convex geometric relation between formula (1) (for the Schubert cycles) and formula (2) (for the Demazure characters) mentioned in the introduction. Namely, we construct GK cubes for a given reduced sequence $(\alpha_{i_1},\ldots,\alpha_{i_\ell})$ of simple roots by two different methods. Both constructions are inductive but the first method uses terminal subwords of the word $I=(i_1,i_2,\ldots,i_\ell)$ while the second method uses initial subwords of $I$.

The first method is the same as in [GK], and can also be described within a more general framework of convex geometric Demazure operators [Ki16, Section 4]. Namely, we consider a (possibly virtual) polytope $P\subset\mathbb{R}^\ell$ obtained as

$$P=D_{i_1}D_{i_2}\cdots D_{i_\ell}(p),$$

where $p\in\mathbb{R}^\ell$ and $D_{i_1}$,..., $D_{i_\ell}$ are convex geometric Demazure operators for $G$ (see [Ki16, Section 4.3] for more details). In this case, the intermediate polytopes $D_{i_\ell}(p)$, $D_{i_\ell-1}D_\ell(p)$,..., and $D_{i_2}\cdots D_{i_{\ell-1}}D_{i_\ell}(p)$ are (possibly twisted) GK cubes for terminal subwords $(i_\ell)$, $(i_{\ell-1},i_\ell)$,..., $(i_2,\ldots,i_{\ell-1},i_\ell)$, respectively.

The second method uses push-pull polytopes. Recall that the Bott-Samelson variety $R_I$ can be constructed as a tower of projective line fibrations:

$$\{\mathrm{pt}\}=R_\varnothing\leftarrow R_{(i_1)}\leftarrow R_{(i_1,i_2)}\leftarrow\ldots\leftarrow R_{(i_1,i_2,\ldots,i_{\ell-1})}\leftarrow R_I.$$

This tower is used in one of the proofs of formula (1) (see [D] or [Ma, Theorem 3.6.18]). Every variety in this tower is obtained from the previous variety as the projectivization of a rank two vector bundle. Using push-pull polytopes we will turn a tower of Bott–Samelson varieties into a tower of convex polytopes:

$$\{p\} = P_{\varnothing} \leftarrow P_{(i_1)} \leftarrow P_{(i_1,i_2)} \leftarrow \ldots \leftarrow P_{(i_1,i_2,\ldots,i_{\ell-1})} \leftarrow P_I.$$

We show that the intermediate polytopes in this tower can be identified with GK cubes for initial subwords $(i_1)$, $(i_1, i_2)$,..., and $(i_1, i_2, \ldots, i_{\ell-1})$, respectively. In particular, we will show that the resulting (always true) polytope $P_I$ is analogous to the virtual polytope $P$. This observation exhibits a relation beween formulas (1) and (2).

We now describe construction of $P_I$ in more detail. We first construct a family of polytopes $P_I(A)$ in convex geometric terms and then explain the relation to Bott–Samelson varieties and GK cubes. Let $(\beta_1,\ldots, \beta_\ell)$ be a collection of vectors in $\mathbb{R}^\ell$ (it is possible that $\beta_i = \beta_j$ for $i \neq j$). Fix a positive definite inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^\ell$, and define the linear function $(\cdot, \beta_i)$ on $\mathbb{R}^\ell$ by the formula:

$$(\alpha, \beta_i) = 2\frac{\langle\alpha, \beta_i\rangle}{\langle\beta_i, \beta_i\rangle}.$$

Denote by $s_{\beta_i}$ the reflection through the hyperplane orthogonal to $\beta_i$, i.e.,

$$s_{\beta_i} : \mathbb{R}^\ell \to \mathbb{R}^\ell; \quad s_{\beta_i} : \alpha \mapsto \alpha - (\alpha, \beta_i)\beta_i.$$

Let $(x_1, \ldots, x_\ell)$ be coordinates on $\mathbb{R}^\ell$. Define $(\ell - 1)$ linear functions on $\mathbb{R}^\ell$:

$$f_1 = (\beta_1, \beta_2)x_1; \quad f_2 = (\beta_1, \beta_3)x_1 + (\beta_2, \beta_3)x_2; \ \ldots \ ;$$

$$f_{\ell-1} = (\beta_1, \beta_\ell)x_1 + (\beta_2, \beta_\ell)x_2 + \ldots + (\beta_{\ell-1}, \beta_\ell)x_{\ell-1}.$$

In particular, $f_i$ depends only on $x_1,\ldots, x_i$. With a collection $A = (a_1, \ldots, a_\ell)$ of sufficiently big real numbers we associate the polytope $P_I(A)$ given by inequalities:

$$0 \leq x_1 \leq a_1; \quad 0 \leq x_2 \leq -f_1(x_1) + a_2; \quad 0 \leq x_3 \leq -f_2(x_1, x_2) + a_3; \ \ldots \ ;$$

$$0 \leq x_\ell \leq -f_{\ell-1}(x_1, \ldots, x_{\ell-1}) + a_\ell$$

(see [HY16, Definition 2.3] for a precise characterization of sufficiently big $A$ in the setting of GK cubes). The polytope $P_I(A)$ is a combinatorial cube and can be thought of as a multidimensional version of a trapezoid. Clearly, $P_I(A) + P_I(B) = P_I(A + B)$, so the space $V_I$ of all virtual polytopes analogous to $P_I(A)$ modulo parallel translations is isomorphic to $\mathbb{R}^\ell$.

For any vector $\lambda \in \mathbb{R}^\ell$, define the vector $A_\lambda \in \mathbb{R}^\ell$ as follows:

$$A_\lambda := ((\lambda, \beta_1), (\lambda, \beta_2), (\lambda, \beta_3), \ldots, (\lambda, \beta_\ell)).$$

Under the isomorphism $\mathbb{R}^\ell \simeq V_I$, the vector $A_\lambda$ corresponds to the (possibly virtual) polytope $P_I(A_\lambda)$. There is a unique vertex $p_I(\lambda)$ of $P_I(A_\lambda)$ such that all coordinates of $p_I(\lambda)$ are nonzero (for generic $\lambda$). It is easy to find these coordinates:

$$p_I(\lambda) = ((\lambda, \beta_1), (\lambda, s_{\beta_1}(\beta_2)), \ldots, (\lambda, s_{\beta_1} \cdots s_{\beta_{\ell-1}}(\beta_\ell))). \tag{5}$$

Note that $p_I(\lambda) = p_I(\mu)$ if and only if $P_I(A_\lambda) = P_I(A_\mu)$.

If $A$ is sufficiently big, then the (true) polytope $P_I(A) \in V_I$ can be constructed inductively using push-pull polytopes. The induction step is based on the following

**Lemma 5.1.** *Let $I^1$, $A$, and $A^1$ denote the sequences $(\beta_2, \ldots, \beta_\ell)$, $(0, a_2, \ldots, a_\ell)$, and $(a_2, \ldots, a_\ell)$, respectively. Choose $\lambda \in \mathbb{R}^\ell$ so that $(\lambda, \beta_1) > 0$. If $P_{I^1}(A^1 + A^1_\lambda)$ and $P_{I^1}(A^1 + A^1_{\lambda-\beta_1})$ are true (not virtual) polytopes then $P_I(A + A_\lambda)$ is analogous to the push-pull polytope $\Delta(P_{I^1}(A^1 + A^1_{\lambda-\beta_1}), P_{I^1}(A^1 + A^1_\lambda))$.*

*Proof.* Consider two parallel facets $\Gamma_0 := P_I(A + A_\lambda) \cap \{x_1 = 0\}$ and $\Gamma_1 := P_I(A + A_\lambda) \cap \{x_1 = (\lambda, \beta_1)\}$ of $P_I(A)$. By construction $\Gamma_0 = P_{I^1}(A^1 + A^1_\lambda) \subset \mathbb{R}^{\ell-1} := \mathbb{R}^\ell \cap \{x_1 = 0\}$. It is easy to check that if we identify $\Gamma_1$ with its projection to $\mathbb{R}^{\ell-1}$ along the $x_1$-axis, then $\Gamma_1 = P_{I^1}(A^1 + A^1_{s_{\beta_1}(\lambda)})$. Indeed, the vertex $p_I(\lambda)$ of $P_I(A_\lambda)$ gets projected to the point

$$((\lambda, s_{\beta_1}(\beta_2)), \ldots, (\lambda, s_{\beta_1} \cdots s_{\beta_{\ell-1}}(\beta_\ell))) \in \mathbb{R}^{\ell-1}.$$

It remains to use that $(\lambda, \mu) = (s_{\beta_1}(\lambda), s_{\beta_1}(\mu))$ for all $\lambda$, $\mu \in \mathbb{R}^\ell$.

Since $s_{\beta_1}(\lambda) = \lambda - (\lambda, \beta_1)\beta_1$, we have $\Gamma_1 = \Gamma_0 - (\lambda, \beta_1)P_{I^1}(A^1_{\beta_1})$. By construction $P_I(A + A_\lambda)$ is the convex hull of parallel facets $\Gamma_0$ and $\Gamma_1$, and the distance between $\Gamma_0$ and $\Gamma_1$ is equal to $(\lambda, \beta_1)$. Hence, $P_I(A + A_\lambda)$ is analogous to $\Delta(P_{I^1}(A^1 + A^1_{\lambda-\beta_1}), P_{I^1}(A^1 + A^1_\lambda))$. □

We now consider the case of Bott–Samelson varieties. Let $(\beta_1, \ldots, \beta_\ell) = (\alpha_{i_\ell}, \ldots, \alpha_{i_1})$, and $\lambda$ a weight of $G$. In what follows, $L_\lambda$ denotes the line bundle on $G/B$ associated with a weight $\lambda$ so that $H^0(L_\lambda, G/B)^*$ for dominant $\lambda$ is isomorphic to the irreducible representation of $G$ with the highest weight $\lambda$. Let $p_I : R_I \to G/B$ denote the projection of the Bott–Samelson variety to the flag variety. By construction $R_I = \mathbb{P}(E)$ where $E$ is a rank two vector bundle over $R_{I^\ell}$. Here $I^\ell$ denotes the sequence $(\alpha_{i_1}, \ldots, \alpha_{i_{\ell-1}})$ (that is, $I^\ell$ coincides with the sequence $(\beta_\ell, \ldots, \beta_2)$). The bundle $E$ can be chosen so that there is a short exact sequence $0 \to \mathcal{O}_{R_{I^\ell}} \to E \to p^*_{I^\ell} L_{\alpha_{i_\ell}} \to 0$. In particular, $c_1(E) = c_1(p^*_{I^\ell} L_{\alpha_{i_\ell}})$ and $c_2(E) = 0$. Hence, if the Khovanskii–Pukhlikov ring $R_{P_{I^\ell}}$ is isomorphic to $CH^*(R_{I^\ell})$ then Theorem 4.1 together with Lemma 5.1 imply the ring isomorphism $R_{P_I} \simeq CH^*(R_I)$.

Note that in order to apply Lemma 5.1 we sometimes have to choose a true polytope $P_{I^1}(A^1 + A^1_\lambda)$ analogous to $P_{I^1}(A^1_\lambda)$ so that $P_{I^1}(A^1 + A^1_\lambda) - P_{I^1}(A^1_{\beta_1})$ is also true. In particular, $P_I(A_\lambda)$ itself is often a virtual (twisted) polytope analogous to the true polytope $P_I(A + A_\lambda)$. In other words, we replace the (not ample) line bundle $p^*_{I^\ell} L_\lambda$ on $R_{I^\ell}$ by a very ample line bundle $L$ such that $L \otimes p^*_{I^\ell} L_{-\alpha_{i_\ell}}$ is also very ample.

Hence, the inductive construction of the polytope $P_I(A)$ (for a sufficiently big $A \in \mathbb{R}^\ell$) via a sequence of push-pull polytopes repeats the inductive construction of the Bott–Samelson variety $R_I$. On the other hand, we defined $P_I(A)$ by the explicit system of linear inequalities. Note that while a convex polytope can be defined as the intersection of finitely many halfspaces (that is, the set of solutions of a system of linear inequalities), a virtual polytope *is* by definition a collection of halfspaces itself (that is, a collection of possibly redundant or inconsistent linear inequalities). Hence, we can view the virtual polytope $P_I(A_\lambda)$ as the collection of inequalities for $A = A_\lambda$. Virtual polytopes can be described more geometrically as (invertible) *convex chains*, that is, linear combinations of convex polytopes with integer coefficients (see [KhP92] for benefits of this approach). In [GK], twisted cubes are defined as convex chains, and not as virtual polytopes. However, it is easy to check that the collection of inequalities in the left column of [GK, (2.21)] (with input from [GK, 3.7]) coincides (up to the change of coordinates $x_i \mapsto -x_i$) with the collection of inequalities, which *is* the virtual polytope $P_I(A_\lambda)$.

*Remark* 5.2. In the case of Bott–Samelson varieities, coordinates of $p_I(\lambda)$ computed in (5) are exactly the coefficients in the Chevalley–Pieri formula in the Chow ring $CH^*(R_I)$. Recall that the Picard group of $R_I$ is spanned by the classes of hypersurfaces $R_{I^j}$ where $I^j$ now denotes the sequence $(i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_\ell)$. The Chevalley–Pieri formula for Bott–Samelson varieties yields the following decomposition of $c_1(p_I^*L_\lambda)$ in the basis $([R_{I^j}])_{1\le j\le \ell}$:

$$c_1(p_I^*L_\lambda) = \sum_{j=1}^{\ell} (\lambda, s_{\beta_1} \cdots s_{\beta_{j-1}}(\beta_j))[R_{I^j}].$$

In particular, the induction step of Lemma 5.1 in this setting can be viewed as a convex geometric counterpart of [D, Proposition 2.1].

On the other hand, there is a simple convex geometric analog of Chevalley–Pieri formula in the Khovanskii–Pukhlikov ring of $P_I$. Let $\Gamma_j$ denote the facet of $P_I(A)$ given by equation $x_j = 0$. The polytope $P_I(A)$ is simple so $\Gamma_j$ defines an element $\partial_{\Gamma_j} \in R_{P_I}$ such that $\partial_{\Gamma_j}(\mathrm{vol}_{P_I(A)}) = \mathrm{vol}_{\Gamma_j}$. Note that the definition of $\partial_{\Gamma_j}$ does not depend on the choice of $A$ whenever $A$ is sufficiently large. Then (5) implies immediately the following identity:

$$\partial_{P_I(A_\lambda)} = \sum_{j=1}^{\ell} (\lambda, s_{\beta_1} \cdots s_{\beta_{j-1}}(\beta_j))\partial_{\Gamma_j}.$$

This formula follows easily from the geometric fact that the volume of $P_I(A)$ is equal to the sum of volumes of $\ell$ pyramides. All pyramides have the common apex, namely, the only vertex of $P_I(A)$ whose coordinates are all nonzero, and the base of the $j$-th pyramide is $\Gamma_j$.

5.2. **Minkowski sums of FFLV polytopes.** Construction of convex geometric push-pull operators is mainly motivated by study of Newton–Okounkov convex bodies of Bott-Samelson varieties. Namely, polytopes that arise as conjectural Newton–Okounkov polytopes can sometimes be also constructed by iterated application of push-pull operators. In such cases, properties of Khovanskii–Pukhlikov rings of push-pull polytopes (Theorem 4.1) allow us to prove that we indeed constructed the full Newton–Okounkov polytope (the same kind of argument was used in Section 5.1). Below we illustrate this method in the case of Bott–Samelson variety

$R_{(1,2,1,3,2)}$ in type $A_3$. This is a particular case of Bott–Samelson varieties in type $A_n$ considered in [Ki18].

In future, we plan to use the same method to complete [Ki18], and prove directly that Minkowski sums of FFLV polytopes yield Newton–Okounkov polytopes of Bott–Samelson varieties in type $A_n$. In [Ki18], we used convex geometric Demazure operators in order to compare volumes, however, this was quite a roundabout way since FFLV polytopes themselves can not be constructed using Demazure operators. On the other hand, push-pull operators seem to be a natural tool to construct FFLV polytopes and their Minkowski sums.

From a geometric viewpoint, points of the Bott–Samelson variety $R_5 := R_{(1,2,1,3,2)}$ can be identified with configurations $(a_1, l_1, a_2, \Pi, l_2)$ of subspaces in $\mathbb{P}^3$ such that $a_1$, $a_2$ are points, $l_1$, $l_2$ are lines, $\Pi$ is a plane, and

$$a_1 \in l_0, l_1;\ a_2 \in l_1 \subset \Pi_0;\ a_2 \in l_2 \subset \Pi,$$

where $l_0 \subset \Pi_0$ are fixed line and plane in $\mathbb{P}^3$. The intermediate Bott–Samelson varieties $R_1 := R_{(1)}$, $R_2 := R_{(1,2)}$, $R_3 := R_{(1,2,1)}$, $R_4 := R_{(1,2,1,3)}$ can be identified with the subvarieties of $R_5$, namely, define $R_4 \subset R_5$ as the hypersurface given by the condition $\{l_2 = l_1\}$. Similarly, define $R_3 \subset R_4$, $R_2 \subset R_3$, and $R_1 \subset R_2$ by the conditions $\Pi = \Pi_0$, $a_2 = a_1$, and $l = l_0$, respectively. Clearly, $R_i$ is the projectivization of a quotient tautological rank two bundle $E_{i-1}$ over $R_{i-1}$ for $i = 1, \dots, i = 5$. For instance, $R_5 = \mathbb{P}(E_4)$ where the fiber of $E_4$ over a point $(a_1, l_1, a_2, \Pi, l_2)$ is the quotient $V(\Pi)/V(a_2)$ (by $V(P)$ we denote the vector subspace whose projectivization is the projective subspace $P$). In particular, the Chow ring of $R_5$ is generated by the first Chern classes $\xi_1, \dots, \xi_5$ of line bundles $\mathcal{O}_{E_0}(1), \dots, \mathcal{O}_{E_4}(1)$. Note also that $R_1 \simeq \mathbb{P}^1$, and $R_2$ is the blow up of $\mathbb{P}^2$ at one point.

We now construct a sequence of polytopes $\Delta_0 := \{0\}$, $\Delta_1 := \Delta_{(1)}$, $\Delta_2 := \Delta_{(1,2)}$, $\Delta_3 := \Delta_{(1,2,1)}$, $\Delta_4 := \Delta_{(1,2,1,3)}$, $\Delta_5 := \Delta_{(1,2,1,3,2)}$ so that every polytope in the sequence is obtained as a push-pull polytope from the previous one. Using the sequence of push-pull polytopes $\Delta_1, \dots, \Delta_5$ and Theorem 4.1 we show that the Khovanskii–Pukhlikov ring of $\Delta_5$ is isomorphic to the Chow ring of $R_5$. More precisely, we construct families of analogous polytopes at each step. In particular, we consider the family of segments $\Delta_1(a) = \{u_1 \in \mathbb{R} \mid 0 \le u_1 \le a\}$ and the family of trapezoids $\Delta_2(a, b) = \{(u_1, u_2) \in \mathbb{R}^2 \mid 0 \le u_1;\ 0 \le u_2 \le b;\ u_1 + u_2 \le a + b\}$.

To construct $\Delta_3$ first note that the Chow ring of $R_3$ is isomorphic to

$$\mathbb{Z}[\xi_1, \xi_2, \xi_3]/(\xi_1^2,\ \xi_2^2 - \xi_1\xi_2,\ \xi_3^2 + \xi_2\xi_3 + \xi_2^2)$$

(this is easy to compute by applying repeatedly projective bundle formula). Hence, $CH^*(R_3)$ is isomorphic to the Khovanskii–Pukhlikov ring of the push-pull polytope $\Delta_3(a, b, c) \subset \mathbb{R}^3$ given by inequalities

$$0 \le u_1 \le a + b;\ 0 \le u_3 \le c;\ 0 \le u_2;\ u_2 + u_3 \le b + c;\ u_1 + u_2 + u_3 \le a + b + c.$$

The ring isomorphism takes $\xi_1$ to $\partial_a$, $\xi_2$ to $\partial_b$ and $\xi_3$ to $\partial_c - \partial_b$. Indeed, if we put $s = c$ in Example 4.5 we get $\Delta_3(a, b, c)$.

It remains to construct $\Delta_4$ from $\Delta_3$ and $\Delta_5$ from $\Delta_4$. Again by projective bundle formula we have the ring isomorphisms:

$$CH^*(R_4) = CH^*(R_3)[\xi_4]/(\xi_4^2 - \xi_2\xi_4);$$
$$CH^*(R_5) \simeq CH^*(R_4)[\xi_5]/(\xi_5^2 - \xi_5(\xi_3 + \xi_4 - \xi_2) + \xi_3(\xi_4 - \xi_2)).$$

Hence, if we take $\Delta_4 = \Delta(P, Q_{\mathcal{F}})$, where $P = \Delta_3(a, b, c)$ and $Q_{\mathcal{F}} = \widehat{P} = \Delta_3(a, b + 1, c))$ we get a polytope whose Khovanskii–Pukhlikov ring is isomorphic to $CH^*(R_4)$. The ring isomorphism sends $\xi_4$ to $\partial_d$. There is a family of polytopes $\Delta_4(a, b, c, d)$ analogous to $\Delta_4$ defined by inequalities:

$$0 \le u_1, u_2, u_3, u_4;\ u_3 \le c;\ u_4 \le d;\ u_1 + u_4 \le a + b + d;$$
$$u_2 + u_3 + u_4 \le b + c + d;\ u_1 + u_2 + u_3 + u_4 \le a + b + c + d.$$

To construct $\Delta_5$ as a push-pull polytope over $\Delta_4$ we first rewrite the relation $\xi_5^2 - \xi_5(\xi_3 + \xi_4 - \xi_2) + \xi_3(\xi_4 - \xi_2))$ (modulo the other relations) in $CH^*(R_5)$ making the change of variable $x = \xi_4 + \xi_5$. We get the ring isomorphism

$$CH^*(R_5) = R_{\Delta_4}[x]/(x^2 - (\partial_c + \partial_d)x + (\partial_c^2 + \partial_d^2)).$$

This gives us a hint on how to choose $\widehat{P}$ and $\mathcal{F}$ in order to construct $\Delta_5$ as $\Delta(P, Q_{\mathcal{F}})$ for $P = \Delta_4(a, b, c, d)$. Namely, $\widehat{P} = \Delta(a, b, c + 1, d + 1)$, and

$$\mathcal{F} = \{\Gamma_{124} \cap \Gamma_4^0,\ \Gamma_{234} \cap \Gamma_3^0,\ \Gamma_{234} \cap \Gamma_4^0,\ \Gamma_{1234} \cap \Gamma_3^0,\ \Gamma_{1234} \cap \Gamma_4^0\},$$

where $\Gamma_i^0$ denotes the facet $\{u_i = 0\}$, and $\Gamma_{124}$, $\Gamma_{234}$ and $\Gamma_{1234}$ denote the facets $\{u_1+u_2+u_4 = a+b+d\}$, $\{u_2+u_3+u_4 = b+c+d\}$ and $\{u_1+u_2+u_3+u_4 = a+b+c+d\}$, respectively. It is not hard to check that $\partial_{\widehat{P}-P} = \partial_c + \partial_d$ and $D_{\mathcal{F}} = \partial_c^2 + \partial_d^2$. Applying Theorem 4.1 we get that the ring $CH^*(R_5)$ is isomorphic to the Pukhlikov–Khovanskii ring of the polytope $\Delta_5(a, b, c, d, e) \subset \mathbb{R}^5$ given by 16 inequalities:

$$u_1, u_2, u_3, u_4, u_5 \ge 0;\ u_1 \le a + b + d;\ u_5 \le e;\ u_3 + u_5 \le c + e;\ u_4 + u_5 \le d + e;$$
$$u_1 + u_4 + u_5 \le a + b + d + e;\ u_2 + u_3 + u_5 \le b + c + d + e;\ u_2 + u_4 + u_5 \le b + c + d + e;$$
$$u_1 + u_2 + u_3 + u_5 \le a + b + c + d + e;\ u_1 + u_2 + u_4 + u_5 \le a + b + c + d + e;$$
$$u_2 + u_3 + u_4 + 2u_5 \le b + c + d + 2e;\ u_1 + u_2 + u_3 + u_4 + 2u_5 \le a + b + c + d + 2e.$$

The isomorphism takes $\xi_5$ to $\partial_e - \partial_d$.

*Remark* 5.3. Note that condition (4) of Theorem 4.1 is nontrivial in this case. We checked it by computing the polynomial $q(s, \Delta_4(a, b, c, d))$. It will be interesting to verify condition (4) in a more geometric way.

It is easy to check that the polytope $\Delta_5$ is equal to the Minkowski sum of FFLV polytopes $P_1(0, a)$, $P_2(0, b, b + c)$, $P_3(0, d, d + e, d + e)$ where the FFLV polytope $P_{n-1}(\lambda_1, \ldots, \lambda_n) \subset \mathbb{R}^{\frac{n(n-1)}{2}}$ is defined by the inequalities $u_k \ge 0$ and

$$\sum_{k \in D} u_k \le \lambda_j - \lambda_i$$

for all Dyck paths $D$ going from $\lambda_i$ to $\lambda_j$ in table $(FFLV)$ where $1 \le i < j \le n$ (below we use only cases $n = 2, 3$ and 4):

$$\begin{array}{ccccccccccc}
\dots & & \lambda_4 & & \lambda_3 & & \lambda_2 & & \lambda_1 & & \\
& \dots & & u_6 & & u_3 & & u_1 & & & \\
& & \dots & & u_5 & & u_2 & & & . & \\
& & & \dots & & u_4 & & & & & \\
& & & & \iddots & & & & & &
\end{array} \qquad (FFLV)$$

As a corollary, we get the following

**Proposition 5.4.** *The normalized volume polynomial of the Minkowski sum $P_1(0, a) + P_2(0, b, b + c) + P_3(0, d, d + e, d + e)$ of FFLV polytopes is equal to the self-intersection index of the divisor $a\xi_1 + (b + c)\xi_2 + c\xi_3 + (d + e)\xi_4 + e\xi_5$ on $R_{12132}$ if both are considered as polynomials in $a$, $b$, $c$, $d$ and $e$.*

We conclude that $\Delta_5(a, b, c, d, e)$ is the Newton–Okounkov polytope $\Delta_v(R_5, \mathcal{O}_{E_1}(a) \otimes \mathcal{O}_{E_2}(b + c) \otimes \mathcal{O}_{E_3}(c) \otimes \mathcal{O}_{E_4}(d + e) \otimes \mathcal{O}_{E_5}(e))$, where $v$ is the valuation considered in [Ki18].

5.3. **Concluding remarks.** As we mentioned in Section 3.2, the projective bundle $\mathbb{P}(E)$ does not change when we replace $E$ with $E \otimes M$ for a line bundle $M$. However, the Chern classes of $E \otimes M$ are not the same as those of $E$. In particular, the same variety $Y = \mathbb{P}(E) = \mathbb{P}(E \otimes M)$ might be associated with combinatorially different push-pull polytopes. In Section 5.1, we fixed a choice of $E$ by the condition $c_2(E) = 0$. In this case, the push-pull polytope $\Delta(P, Q_{\mathcal{F}})$ has a particularly simple combinatorial structure, since $\mathcal{F} = \{\varnothing\}$. However, what we gained in simplicity we lost in positivity as the resulting polytopes do not degenerate to Newton–Okounkov polytopes of flag varieties. Instead they turn into twisted cubes, which are not true polytopes (cf. [GK, Remark 3.21], [HY16, Examples 4.1–4.3] and [Fu18, Example B1]).

In Section 5.2, we fixed a choice of $E$ by the condition $c_1(E) = c_1(L)$ for a certain line bundle $L$. To choose $L$ we used tacitly an explicit combinatorial rule that can be applied to all initial subwords of $\underline{w_0} = s_1(s_2s_1)\cdots(s_ns_{n-1}\cdots s_1)$. To find $\mathcal{F}$ we realized $c_2(E)$ (considered as an element of Khovanskii–Pukhlikov ring $R_P$) as a sum of codimension two faces of $P$. This is the subtlest choice to make, and here we tacitly used data from the Minkowskii sum of FFLV polytopes $P_1(0, 1) + P_2(0, 1, 2) + \ldots + P_{n-1}(0, 1, \ldots, n)$. We conjecture that this Minkowski sum of FFLV polytopes (and more generally the Minkowski sum of FFLV polytopes used in [Ki18, Theorem 3.1]) can be constructed by successive application of convex geometric push-pull operators (in Section 5.2 we checked this conjecture for all proper initial subwords of $s_1(s_2s_1)(s_3s_2s_1)$). This will allow us to produce Newton–Okounkov polytopes of Bott–Samelson varieties that do not break in the limit when passing to flag varieties.

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