Pictures and definitions

for the talk “Push-pull polytopes and projective line bundles” by Valentina Kiritchenko

Push-pull operators on convex polytopes

arXiv:2012.15690 [math.AG]

Reminder Projective bundle formula:

$$A^*(Y) \simeq A^*(X)[t] \Big/ \left(t^2 - c_1(E)t + c_2(E)\right)$$

where $c_1(E)$, $c_2(E)$ are Chern classes of E,

and $t = c_1(\mathcal{O}_E(1))$.

(One of) definitions of Chern classes

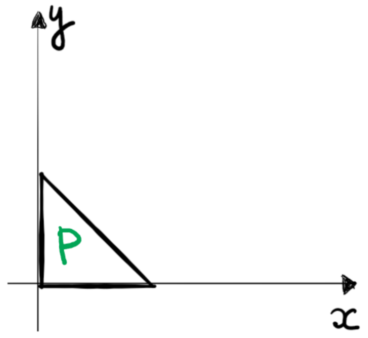
Push-pull polytope

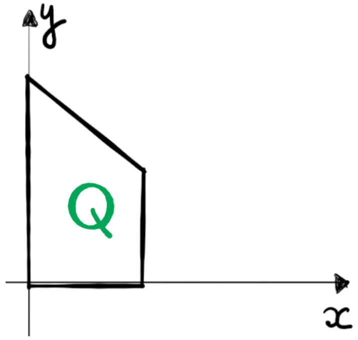
- c_1: $\hat{P}$ is analogous to P
- c_2: Q is a codimension 2 truncation of $\hat{P}$

$$\Downarrow$$

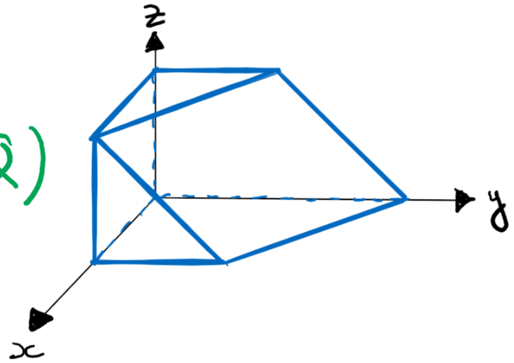
$\Delta(P, Q) =$ Cayley sum of P and Q

$=$ convex hull of $(P \times 0) \cup (Q \times 1) \subset \mathbb{R}^d \times \mathbb{R}$

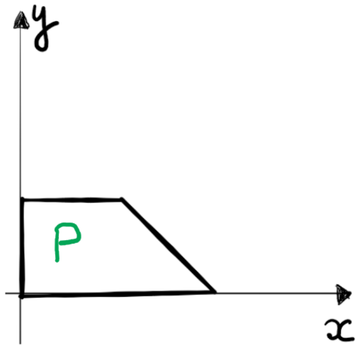


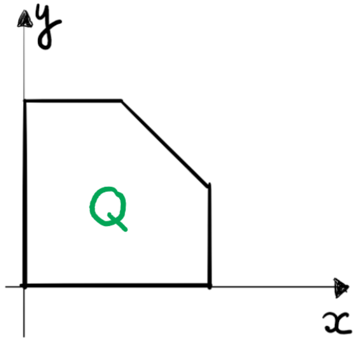


$\Delta(P, Q)$

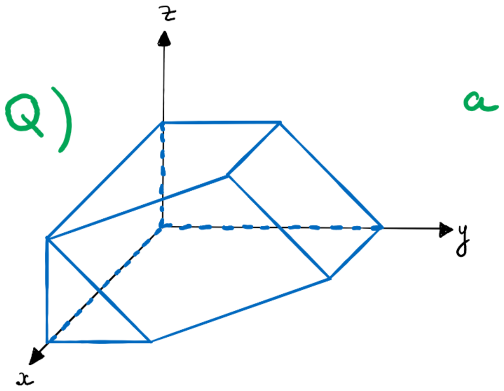


Feigin-Fourier-Littelmann-Vinberg polytope



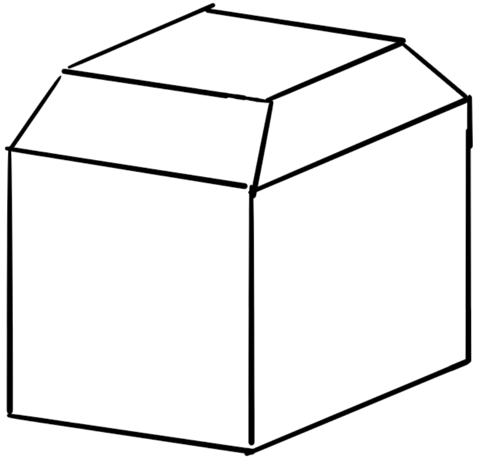


$\Delta(P, Q)$



a Newton-Okounkov polytope of the Bott-Samelson variety R_{121}

Examples of codimension 2 truncations
3D



a truncated cube

Taken from Okounkov's lecture notes

80 K-theoretic computations in enumerative geometry

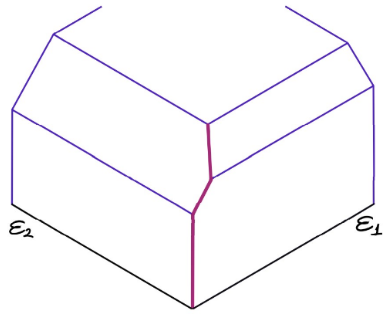


FIGURE 6.4.5. Universal deformation of a curve with two bubbles. The ith node remains intact over $\varepsilon_i = 0$.

another truncated cube

Simple geometric mitosis

Definition 1 $\quad F \subset P \subset \overset{\Delta}{\overset{\shortparallel}{\Delta(P,Q)}}$ - face $\rightsquigarrow$

F is admissible if there exists a unique face $\exp(F) \subset \Delta$ such that $F = \exp(F) \cap P$.

Definition 2 $\quad F$ - admissible, $\dim F = \ell$

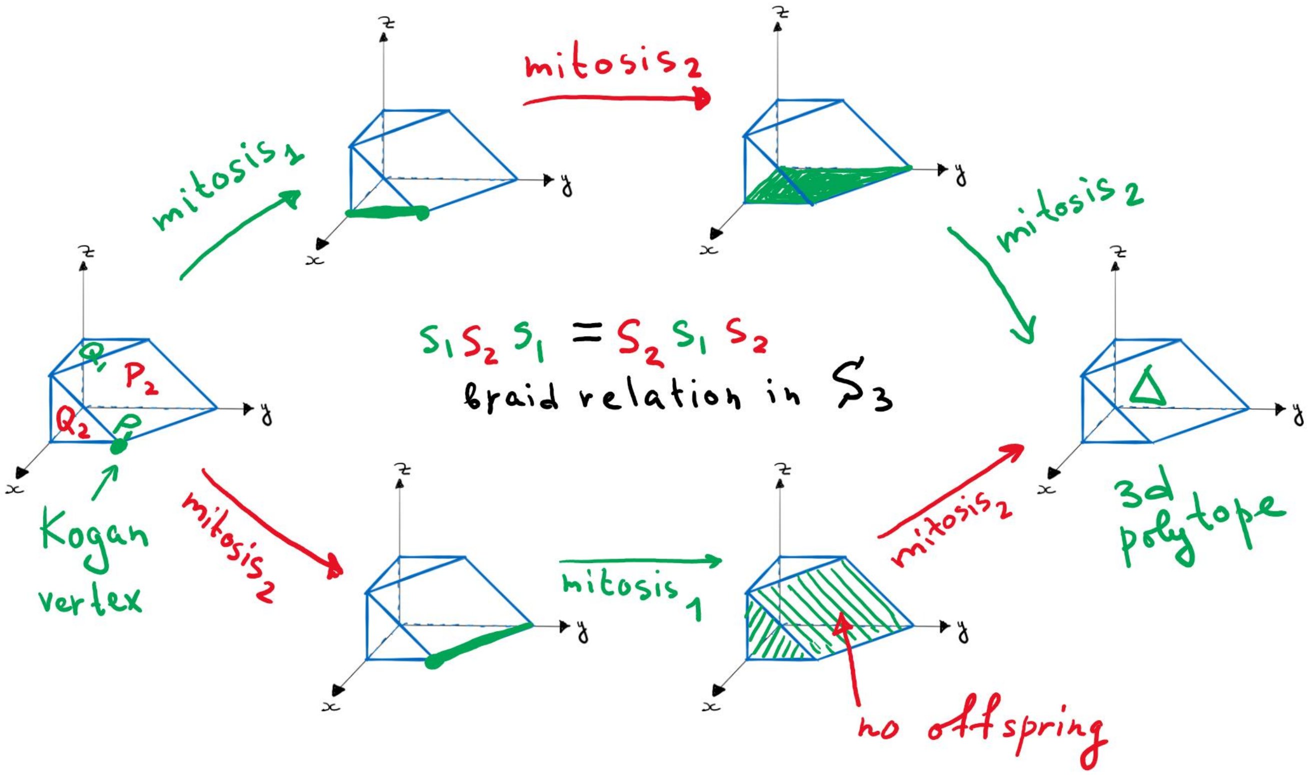
$\rightsquigarrow$ $\text{mitosis}(F) = \{\overset{\exp(F)}{\overset{\shortparallel}{E_1}}, \dots, E_k\}$, $\dim E_i = \ell + 1$,

defined by

(1) $E_i \cap P \neq \emptyset$, $E_i \cap Q \neq \emptyset$ („non-horizontal")

(2) $E_i \cap Q \subseteq \exp(F) \cap Q$ („degenerates to ∂F")

Mitosis in type A_2

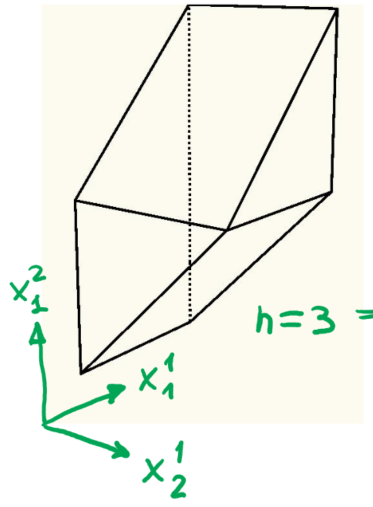


Gelfand - Zetlin polytope (type A_{n-1})

$$\begin{array}{ccccccccc}
\lambda_1 & & \lambda_2 & & \lambda_3 & & \dots & & \lambda_n \\
& x_1^1 & & x_2^1 & & \dots & & x_{n-1}^1 & \\
& & x_1^2 & & x_2^2 & \dots & x_{n-2}^2 & & \\
& & & \ddots & & ⋰ & & & \\
& & & x_1^{n-2} & & x_2^{n-2} & & & \\
& & & & x_1^{n-1} & & & &
\end{array}$$

2d inequalities on $d := \frac{n(n-1)}{2}$ variables

$$\begin{array}{ccc} a & & b \\ & c & \end{array} \iff a \leq c \leq b$$



$n=3 \Rightarrow d=3$

Gelfand-Zetlin polytope (type C_n)

$$
\begin{array}{llllll}
 & \lambda_1 & \lambda_2 & \cdots & \lambda_n & 0 \\
\text{2n-1 variables} \left\{ \begin{array}{l} \\ \\ \end{array} \right. & x_1^1 & x_2^1 & \cdots & x_n^1 & \\
 & & x_1^2 & x_2^2 \;\cdots & x_{n-1}^2 & 0 \\
 & & & \ddots & & \vdots \\
\text{3 variables} \left\{ \begin{array}{l} \\ \\ \end{array} \right. & & & x_1^{2n-3} & x_2^{2n-3} & \\
 & & & & x_1^{2n-2} & 0 \\
 & & & \text{1 variable} \; \{ & x_1^{2n-1} &
\end{array}
$$

2d inequalities on $d := n^2$ variables

Naoki Fujita, Schubert calculus from polyhedral parametrizations of Demazure crystals, arXiv:2008.04599 [math.RT]