Polynomials and Polytopes

Valentina Kiritchenko*

*Faculty of Mathematics and Laboratory of Algebraic Geometry and its applications, HSE University

and

Kharkevich Institute for Information Transmission Problems RAS

Sirius, April 24, 2023

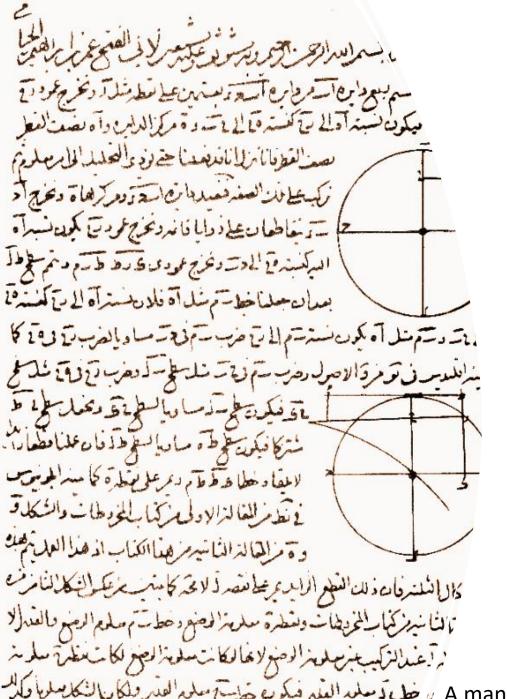
ع اسع ومسمن عل ميكون نسبترا ولل من كسترها للهاشد و ، مركز الدلين وا ، بصف لعط مانا مزل بالديمد فالحقر ووالتحليل تكيسط للالصغر فعددا محاسة وومرزهاة وعزيرا - وخاطعان على دوايا فاندوبخ جمود في كمحل نسباً. السكسندج المحرة ونخرج عودى كارط طرح وتم طحطا بمدار جلناخوس مثل آه فلان ستراه الم م كلنده المكون نستر مالات طرب مع والمساد بالطرب ت والمكا شرادلد مدف تومردالاص رض متم وي مناسط مدوخ ومن تروق شد مح يقفكون عراد المالطية وتعارط ط شتركا فبكون طحطم مساديا تسجط وفان علنا فطعادت لاعقا وخطا هرقوق وترعلى بمطرة كاستراغ مي فينط مرالفا لتزالا وف يتكا للخوطات والشكاق وة مزالمالة الناشيم عدا الكتاب اذ هذا العل مجب كالانتشرقان ذلك السطع الرابر بم عط متصر و لايحد كاستر عرعك التكالنا مرم تاننا سيرتكا بالمخوطات وتشطرة ملمة المصع وحطت معلمه الرص والعدرالا " اعتال كيب بر معذ الوج لا عالكات معد المج لكات مقدة ملامة A manuscript of Omar Khayyam ، خط، دَسل، العند فيكون خاسج سلم العدر ولكاما الشكام

Omar Khayyam constructed roots of a cubic equation by intersecting a circle with a hyperbola.

واسع ومستعط مط ميكون نسبترا وللابنة كنسترق للاستددة مركز الدلين وآه بضع بمت لقومانا بالالمعد الحقر والعدلد تكيسط للالصغر فعددا محاسة وومرزهاة وعزيرا _ مقاطعان على دوايا فانه ديخ جمودي كمحل نسبراً السكسندج المحرة ونخرج عودى كارط طرح وتم طحطا بعدان جلناخو مثل أة فلان ستراة الري كمسدق ٢٠ وتم مند ٦٠ كون نسبة م ١١ ت حرب مود تر مساويل مرب ت ٤٠٠ كا شرادلدر ف تومرد الاصرل دخر - م دير - شل الم - لدعر ، وق شل ع يقفكون عجر لدسا وبالطي يحد وتعل على ط شتركا فبكون طحطم مساديا تسجط وفان علنا فطعادت لاعقا وخطا هرقوق وترعلى بمطرة كاستراغ مي فينط مرالفا لترالا وليمركنا الخوطات والشكل وة مزالمالة الناشيم عنا الكتاب اذهذا العل يجب كالانتنادة للالسطم الالدي عطمعه ولاجر كاست عظالتك للانام تالنا سيركما المخوطات وتسطرة سلمة المصم وحطت مسلم الرجع والعدرالا " اعتال كسبع بمعد الوجولا عالكات مع المجولكات مقدة ملامة A manuscript of Omar Khayyam ، خط، دَسل، العلرفيكر، خاسم العدر ولكامنا لشكام لما فك

Omar Khayyam constructed roots of a cubic equation by intersecting a circle with a hyperbola.

Warning: this is not a solution by radicals.



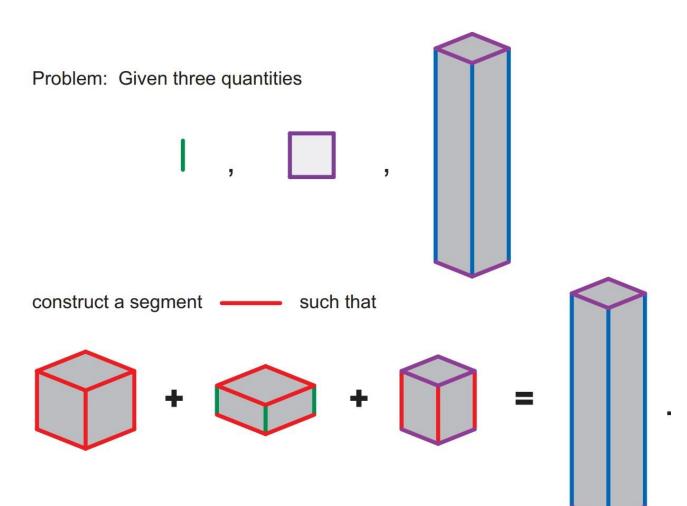
Omar Khayyam constructed roots of a cubic equation by intersecting a circle with a hyperbola.

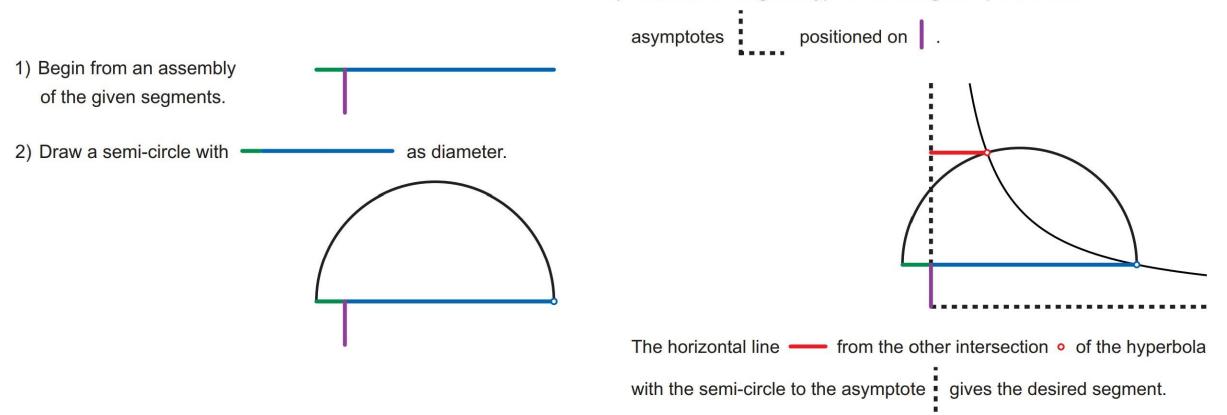
Warning: this is not a solution by radicals.

Question: is it possible to construct the roots of a cubic equation by straightedge and compass?

A manuscript of Omar Khayyam ، خل، دَمل، العند فيكون خاسمَ ملم العدر ولكان الشكاملوا وكل

Modern account of Khayyam's construction by Deborah A. Kent and David J. Muraki (in geometric terms, no algebraic notation).





3) Draw the rectangular hyperbola through the point • with

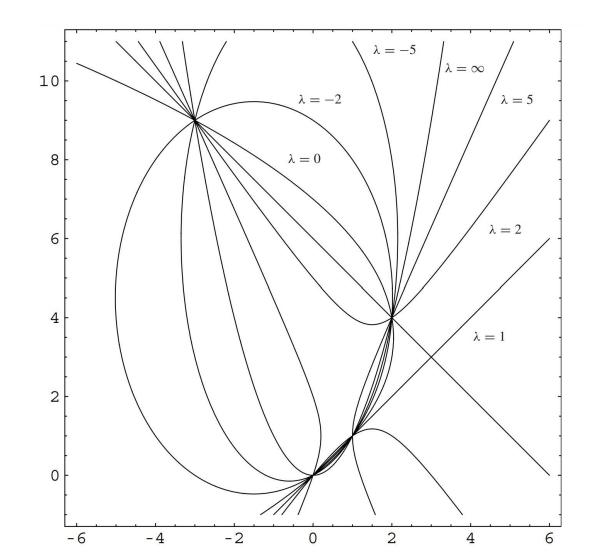
Solving quartic equations via pencils of conics

Pencil spanned by parabola $y = x^2$ and parabola $6x = y^2 - 7y$.

Aukley

Picture from paper "Solving the

Quartic with a Pencil" by Dave



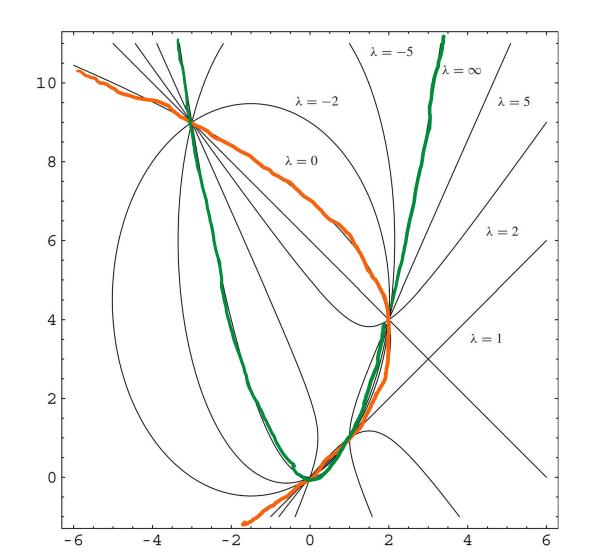
Solving quartic equations via pencils of conics

Pencil spanned by parabola $y = x^2$ and parabola $6x = y^2 - 7y$.

The conic labelled by λ is given by equation:

$$6x - y^2 + 7y + \lambda(y - x^2) = 0$$

Picture from paper "Solving the Quartic with a Pencil" by Dave Aukley



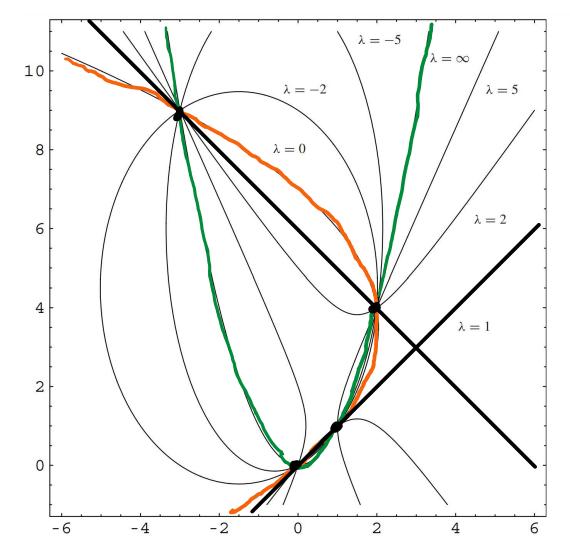
Solving quartic equations via pencils of conics

Pencil spanned by parabola $y = x^2$ and parabola $6x = y^2 - 7y$.

The conic labelled by λ is given by equation:

$$6x - y^2 + 7y + \lambda(y - x^2) = 0$$

There are 3 degenerate conics (=pairs of lines) in the pencil. They can be found by solving a cubic equation.



Question: how to define the degree of a polynomial in two variables?

Example:
$$f(x, y) = 2xy + x^2y + 3x^4y + 5x^2y^2 + xy^3$$

Question: how to define the degree of a polynomial in two variables?

Example:
$$f(x, y) = 2xy + x^2y + 3x^4y + 5x^2y^2 + xy^3$$

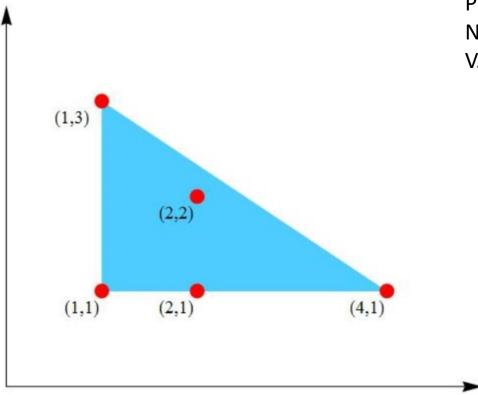
Why bother?

Question: how to define the degree of a polynomial in two variables?

Example:
$$f(x, y) = 2xy + x^2y + 3x^4y + 5x^2y^2 + xy^3$$

Why bother?

Motivation: generalize the Fundamental Theorem of Algebra to two (3, 4,..., n) variables.



Pictures from paper "Ideas of Newton-Okounkov bodies" by V.K., E.Smirnov, V.Timorin

The support (red points) and the Newton polygon (blue surface) of the polynomial $2xy + x^2y + 3x^4y + 5x^2y^2 + xy^3$.

Minding theorem (special case)

Let f(x,y) and g(x,y) be polynomials with the same Newton polygon *P*. Then the system of two equations in two unknowns f(x,y) = g(x,y) = 0

has at most *2area(P)* totally nonzero solutions (whenever the number of solutions is finite).

Minding theorem (special case)

Let f(x,y) and g(x,y) be polynomials with the same Newton polygon *P*. Then the system of two equations in two unknowns f(x,y) = g(x,y) = 0

has at most *2area(P)* totally nonzero solutions (whenever the number of solutions is finite).

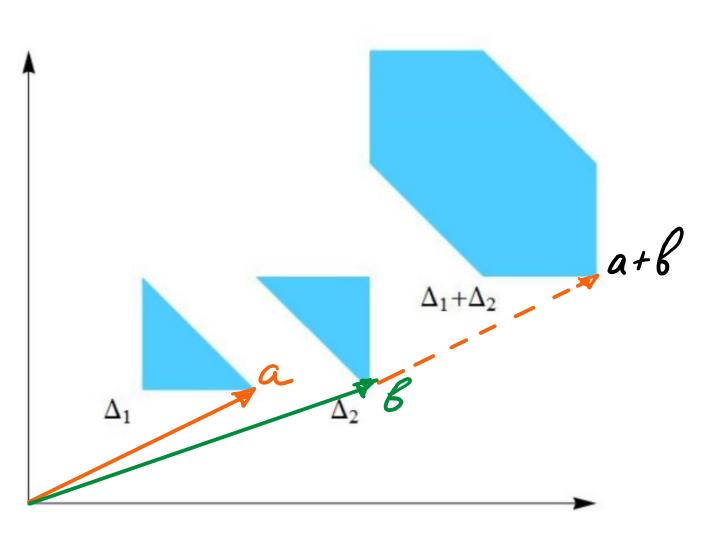
Example:

f(x,y) = axy + bx + cy + d,g(x,y) = a'xy + bx' + c'y + d'

Two hyperbolas with parallel asymptotes intersect at two points.

Minding theorem (general case)

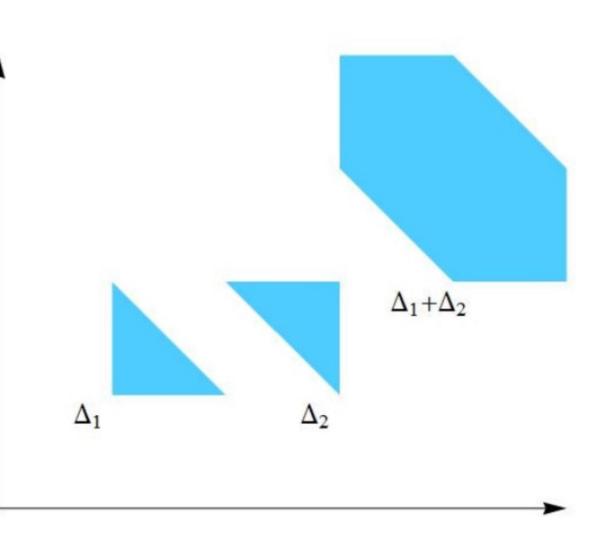
Question: what if f(x,y) and g(x,y) have different Newton polygons P and Q?



Minding theorem (general case)

Question: what if f(x,y) and g(x,y) have different Newton polygons P and Q?

Answer: use Minkowski sum of polygons.



Minding theorem (general case)

Let f(x,y) and g(x,y) be polynomials with the Newton polygons P and Q. Then the system of two equations in two unknowns f(x,y) = g(x,y) = 0

has at most [*area(P+Q)-area(P)-area(Q)*] totally nonzero solutions (whenever the number of solutions is finite).

Minding theorem (general case)

Let f(x,y) and g(x,y) be polynomials with the Newton polygons P and Q. Then the system of two equations in two unknowns f(x,y) = g(x,y) = 0

has at most [*area(P+Q)-area(P)-area(Q)*] totally nonzero solutions (whenever the number of solutions is finite).

Example: the system

 $f(x,y) = a_n x^n + \dots + a_1 x + a_0$ $g(x,y) = a_m y^m + \dots + b_1 y + b_0$

has mn solutions.

Generalizations

Theorems:

- Bernstein-Kushnirenko-Khovanskii theorems (1970s)
- Brion-Kazarnovskii formula (1980s)

Areas of mathematics:

- toric geometry and theory of Newton polytopes
- canonical bases in representation theory
- theory of Newton-Okounkov convex bodies and toric degenerations
- Schubert calculus

Mathematics Genealogy Project

Askold Georgievich Khovanskii

MathSciNet

Ph.D. Steklov Institute of Mathematics 1973

Dissertation: Representability of Function in Quadratures

Advisor: Vladimir Igorevich Arnold

Students: Click <u>here</u> to see the students listed in chronological order.

Name	School	Year D	Descendants
Borodich, Feodor	Lomonosov Moscow State University	1984	
<u>Burda, Yuri</u>	University of Toronto	2012	
Chulkov, Sergey	Stockholm University	2005	
Firsova, Tanya	University of Toronto	2010	
<u>Gelfond, Olga</u>	Lomonosov Moscow State University	1984	
Izadi, Farzali	University of Toronto	2001	6
Kaveh, Kiumars	University of Toronto	2002	3
<u>Kiritchenko,</u> <u>Valentina</u>	University of Toronto	2004	
<u>Mazin, Mikhail</u>	University of Toronto	2010	
Monin, Leonid	University of Toronto	2019	
Soprunov, Ivan	University of Toronto	2002	
Soprunova, Jenya	University of Toronto	2002	
<u>Timorin, Vladlen</u>	Steklov Institute of Mathematics	2003	1
<u>Timorin, Vladlen</u>	University of Toronto	2004	1

Family history

Annals of Mathematics ${\bf 176}$ (2012), 925–978 http://dx.doi.org/10.4007/annals.2012.176.2.5

Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory

By KIUMARS KAVEH and A. G. KHOVANSKII

To the memory of Vladimir Igorevich Arnold

Thank you for your attention!

Slides and references can be found on the webpage: me.hse.ru/valkir