Polynomials and Polytopes

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and

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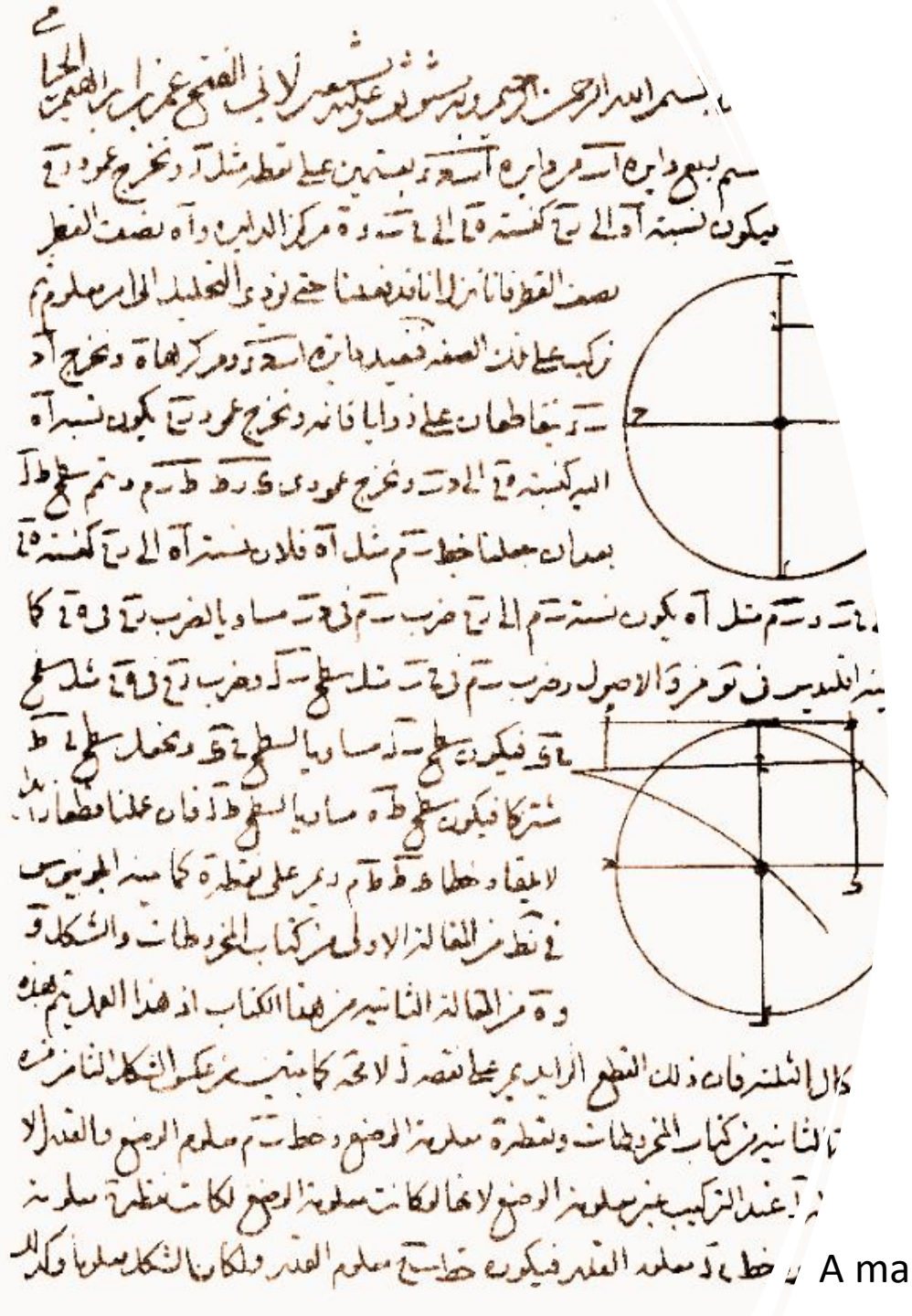
Sirius, April 24, 2023

Solving cubic equations via conics

Omar Khayyam constructed roots of a cubic equation by intersecting a circle with a hyperbola.



A manuscript of Omar Khayyam

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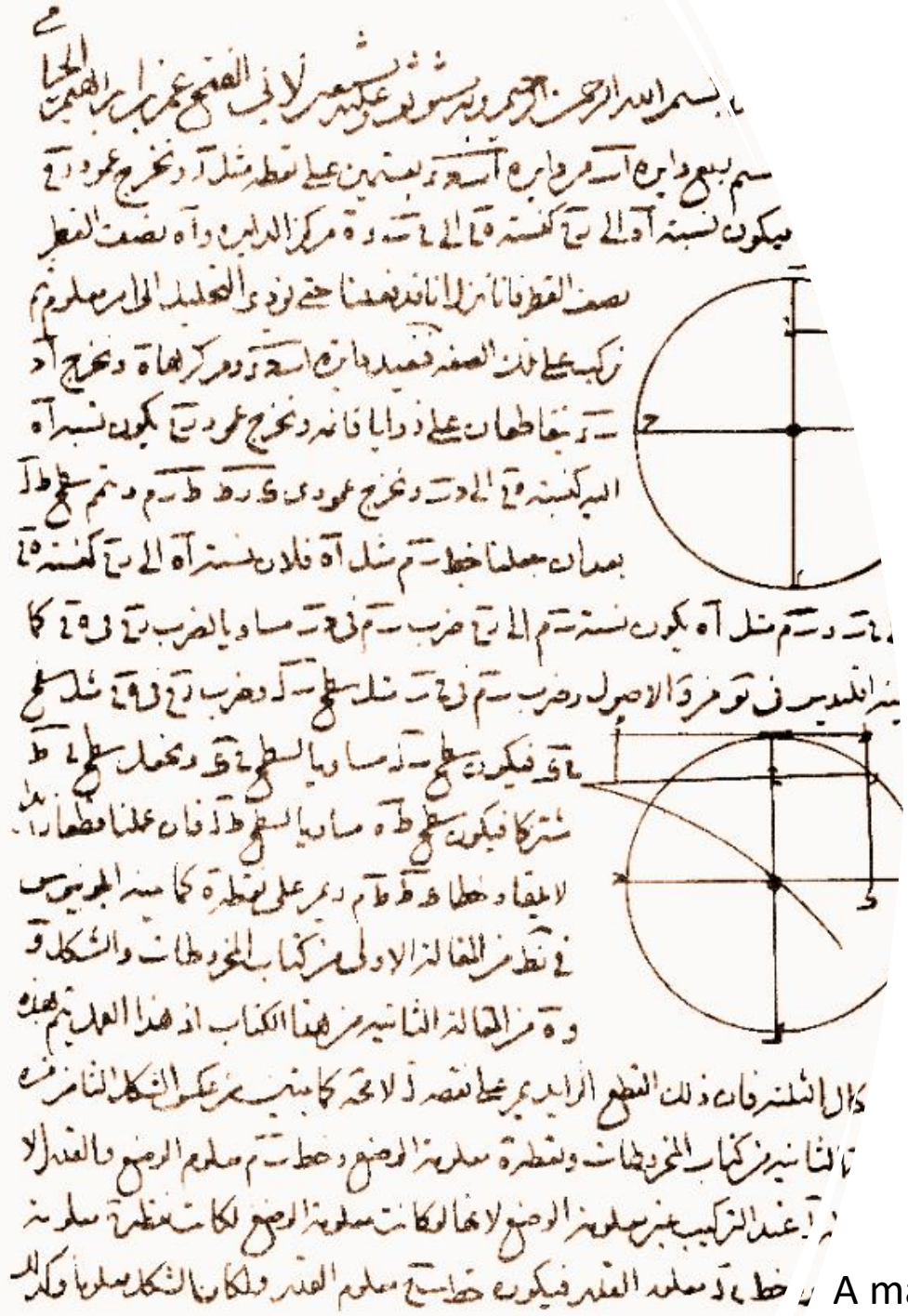

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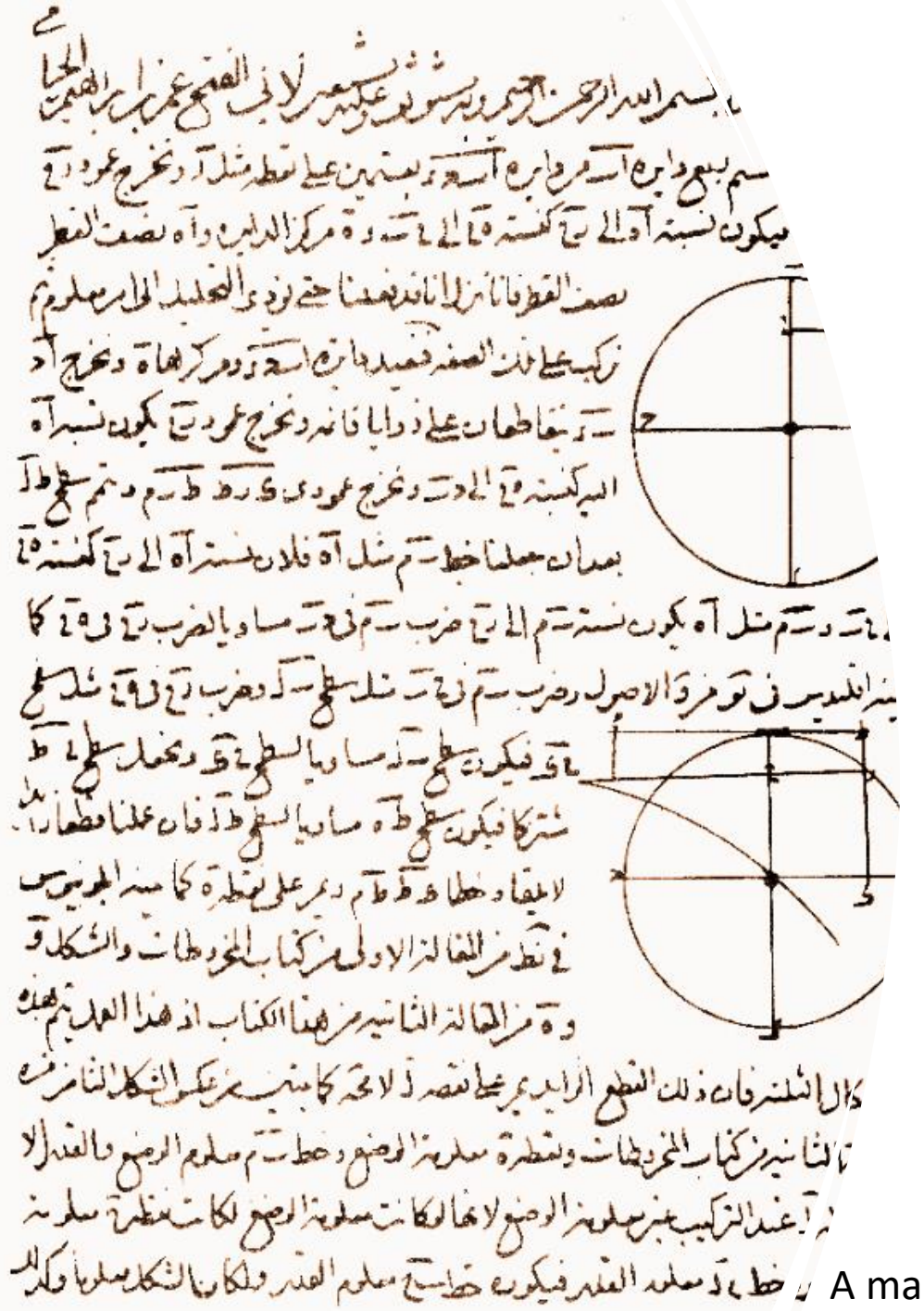

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Solving cubic equations via conics

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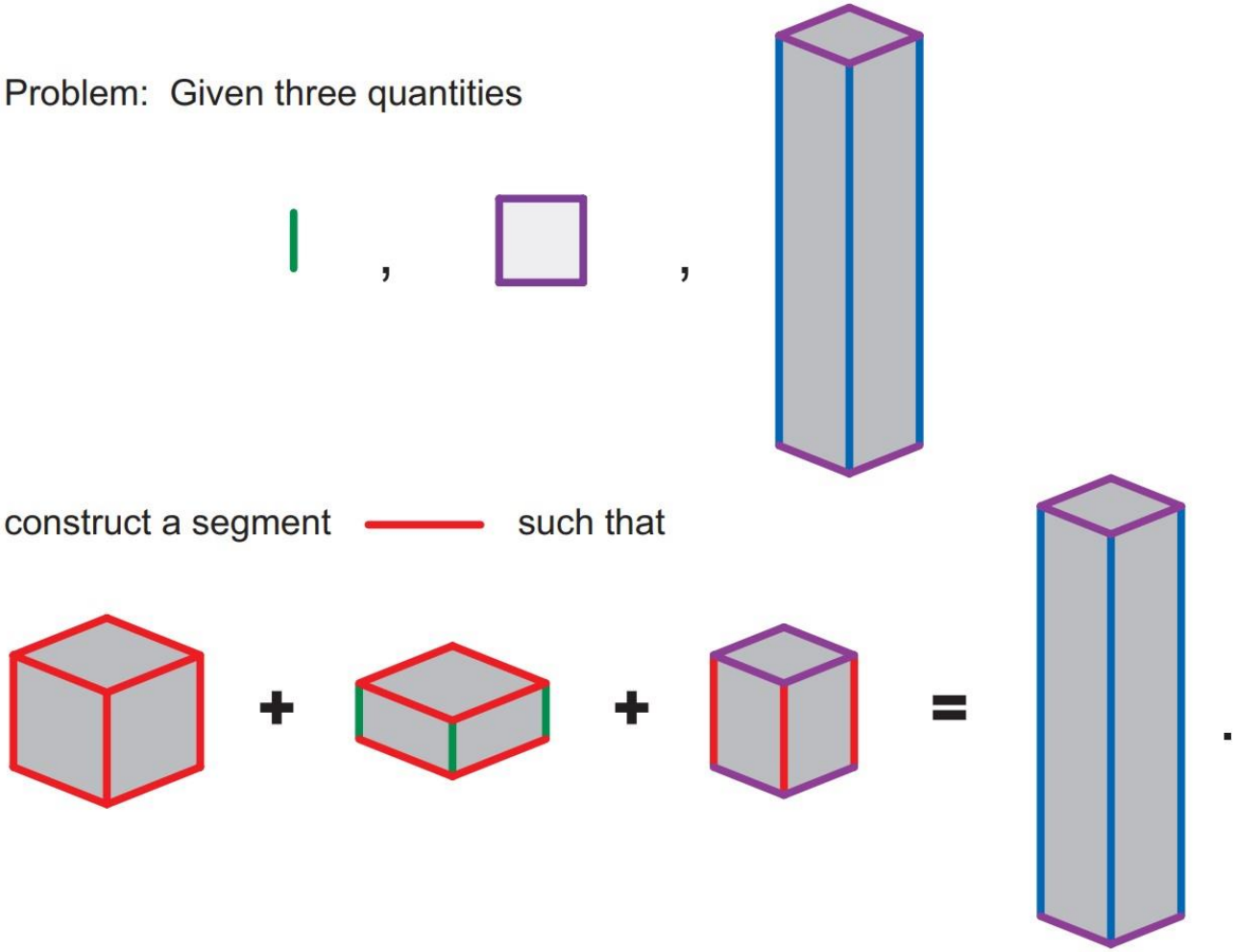
Warning: this is not a solution by radicals.

Question: is it possible to construct the roots of a cubic equation by straightedge and compass?

Solving cubic equations via conics

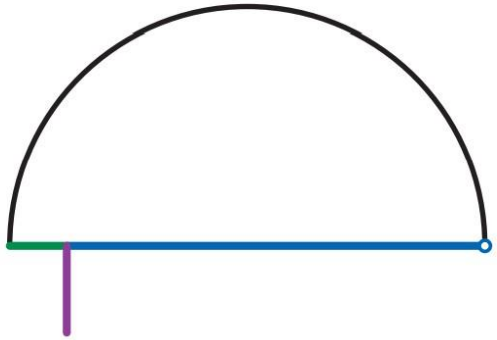
Modern account of Khayyam's construction by Deborah A. Kent and David J. Muraki (in geometric terms, no algebraic notation).



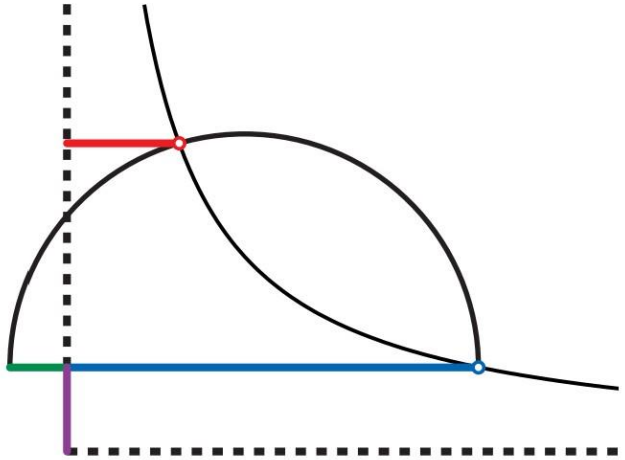
Solving cubic equations via conics

1) Begin from an assembly of the given segments.

2) Draw a semi-circle with as diameter.



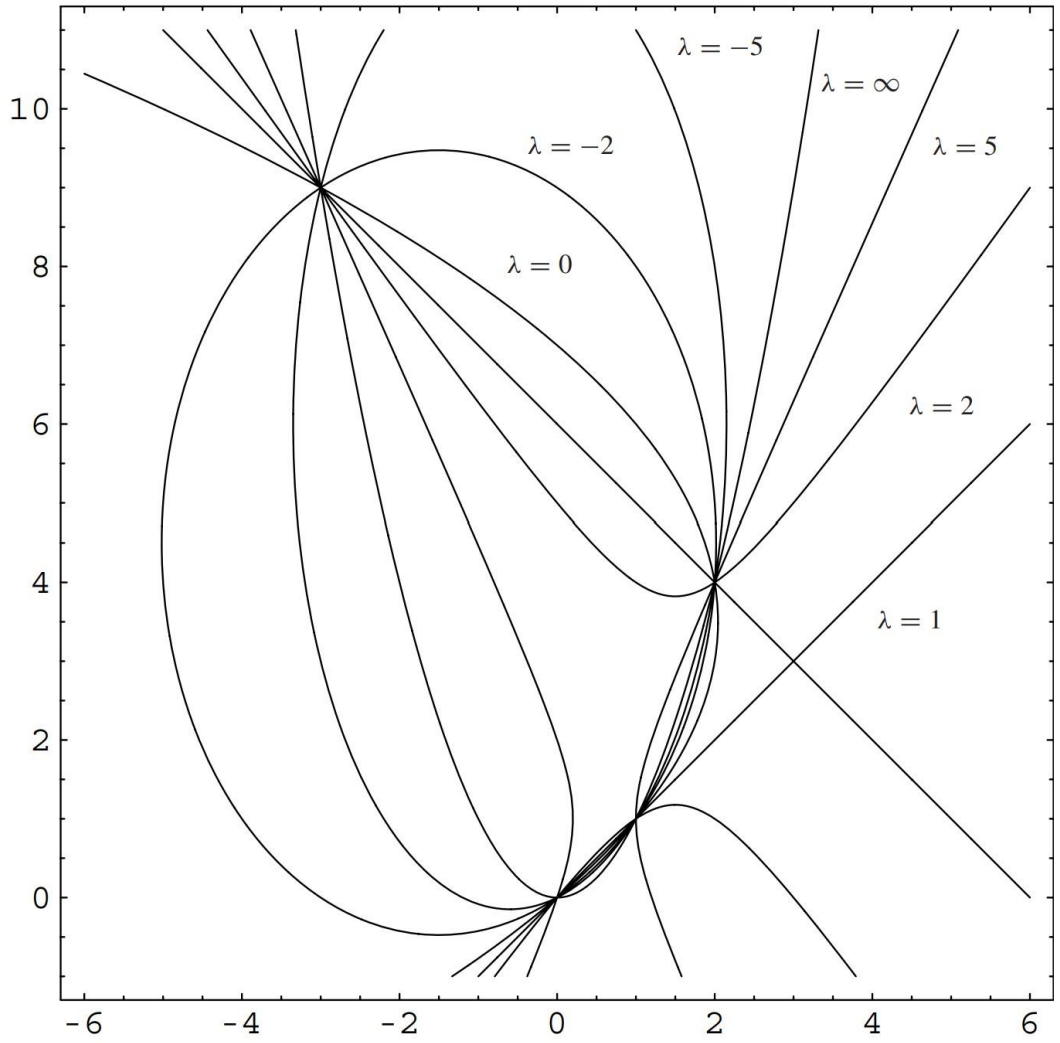
3) Draw the rectangular hyperbola through the point with asymptotes positioned on .



The horizontal line from the other intersection of the hyperbola with the semi-circle to the asymptote gives the desired segment.

Solving quartic equations via pencils of conics

Pencil spanned by parabola $y = x^2$ and parabola $6x = y^2 - 7y$.



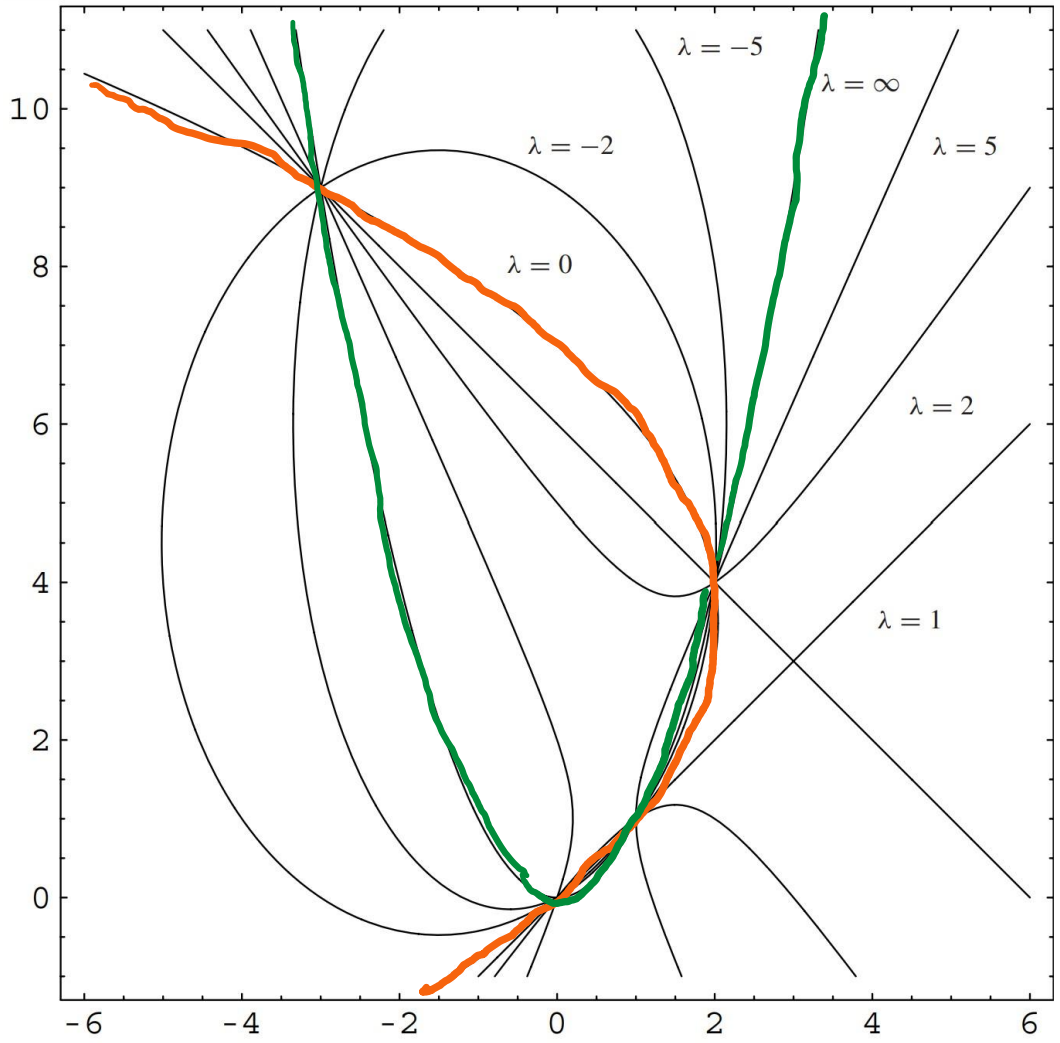
Picture from paper "Solving the Quartic with a Pencil" by Dave Aukley

Solving quartic equations via pencils of conics

Pencil spanned by parabola $y = x^2$ and parabola $6x = y^2 - 7y$.

The conic labelled by λ is given by equation:

$$6x - y^2 + 7y + \lambda(y - x^2) = 0$$



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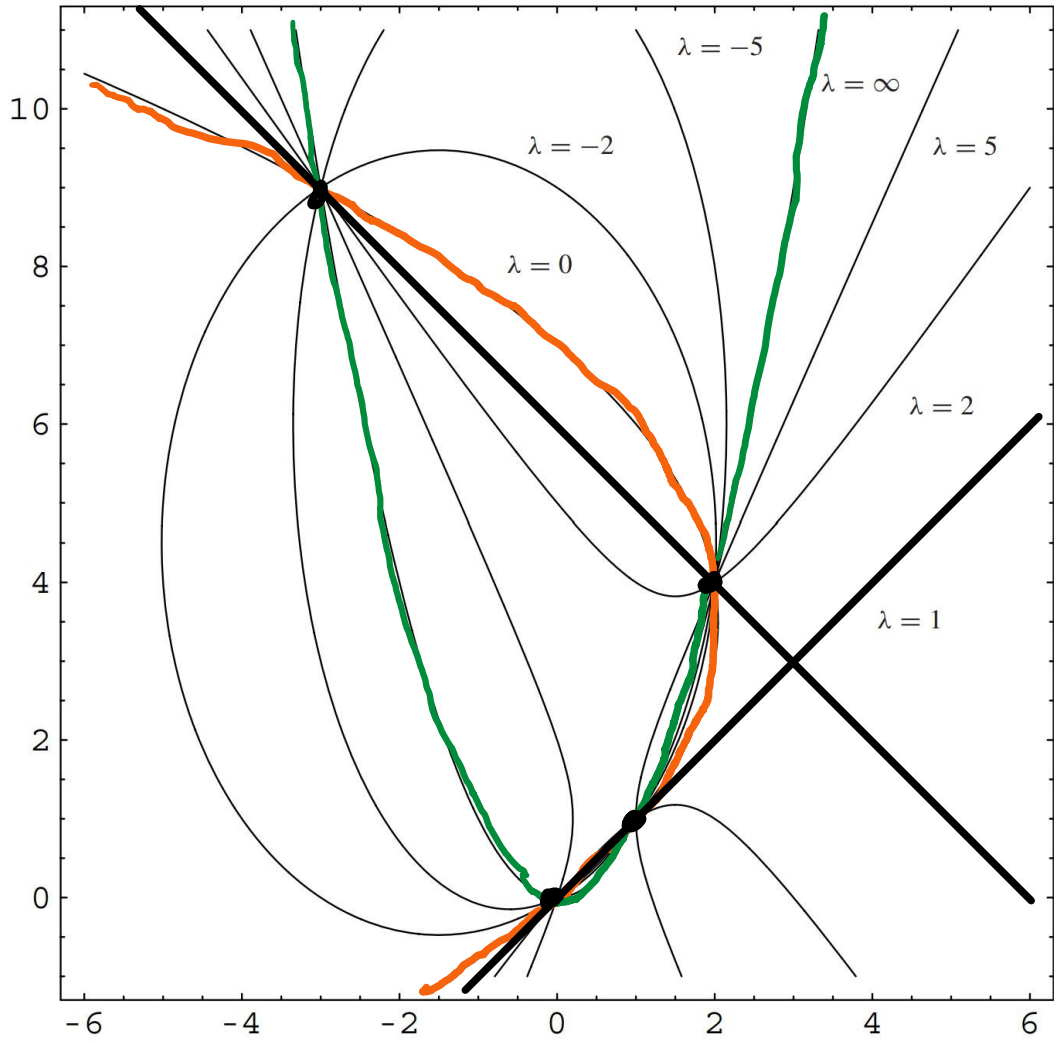
There are 3 degenerate conics (=pairs of lines) in the pencil. They can be found by solving a cubic equation.



Newton polygons

Question: how to define the degree of a polynomial in two variables?

Example: $f(x, y) = 2xy + x^2y + 3x^4y + 5x^2y^2 + xy^3$

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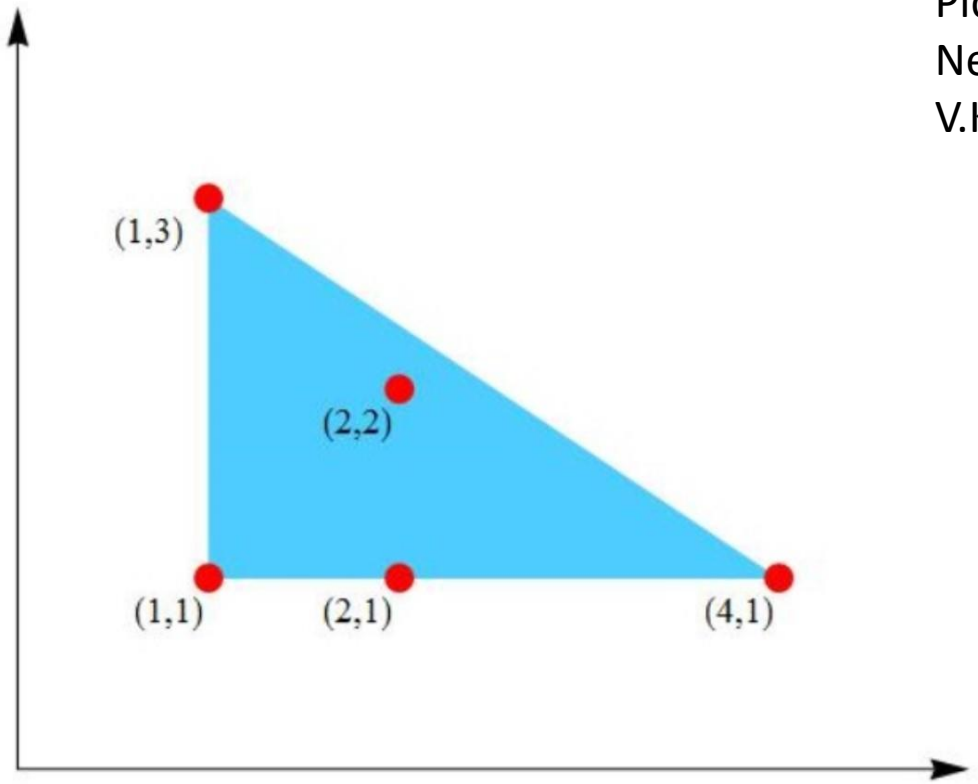
Motivation: generalize the Fundamental Theorem of Algebra to two (3, 4,…, *n*) variables.

Newton polygons

Pictures from paper “Ideas of Newton-Okounkov bodies” by V.K., E.Smirnov, V.Timorin



The support (red points) and the Newton polygon (blue surface) of the polynomial $2xy + x^2y + 3x^4y + 5x^2y^2 + xy^3$.

Minding theorem (special case)

Let *f(x,y)* and *g(x,y)* be polynomials with the same Newton polygon *P.* Then the system of two equations in two unknowns

$$f(x,y) = g(x,y) = 0$$

has at most *2area(P)* totally nonzero solutions (whenever the number of solutions is finite).

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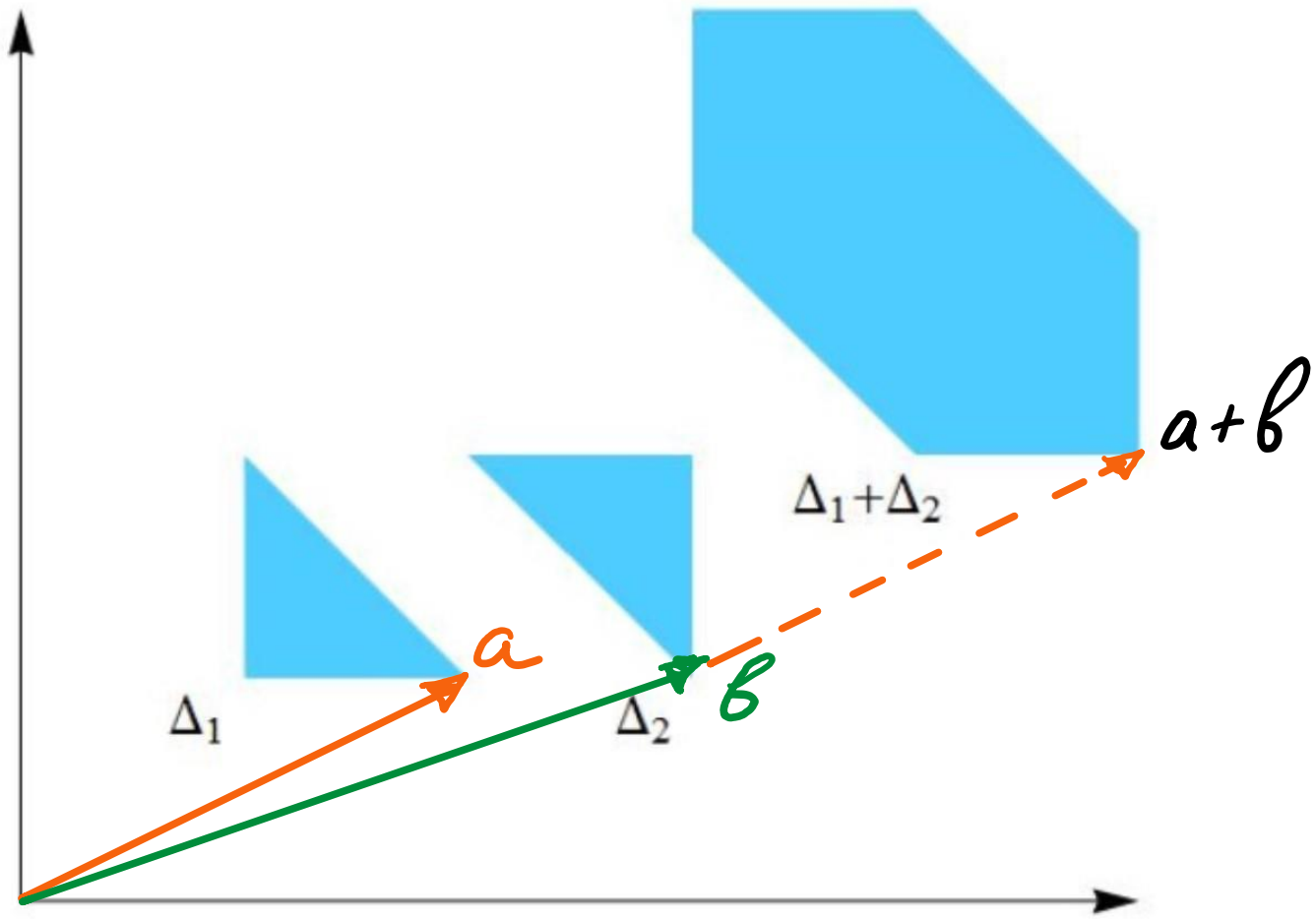
Example:

$$f(x,y) = axy + bx + cy + d,$$
$$g(x,y) = a'xy + bx' + c'y + d'$$

Two hyperbolas with parallel asymptotes intersect at two points.

Minding theorem (general case)

Question: what if f(x,y) and g(x,y) have different Newton polygons P and Q?



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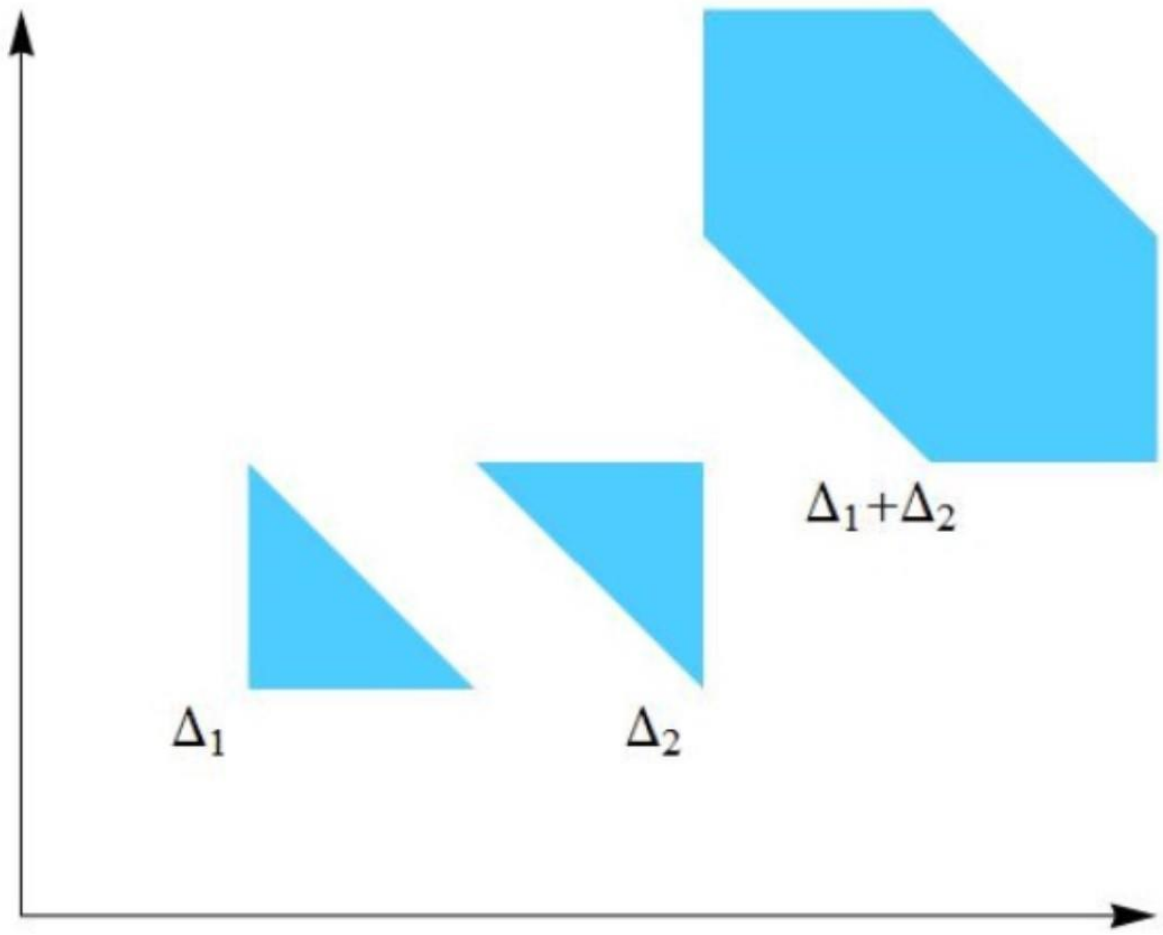
Answer: use Minkowski sum of polygons.



Minding theorem (general case)

Let *f(x,y)* and *g(x,y)* be polynomials with the Newton polygons *P* and *Q.* Then the system of two equations in two unknowns

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Example: the system

$$f(x,y) = a_n x^n + \cdots + a_1 x + a_0$$
$$g(x,y) = a_m y^m + \cdots + b_1 y + b_0$$

has *mn* solutions.

Generalizations

Theorems:

- Bernstein-Kushnirenko-Khovanskii theorems (1970s)
- Brion-Kazarnovskii formula (1980s)

Areas of mathematics:

- toric geometry and theory of Newton polytopes
- canonical bases in representation theory
- theory of Newton-Okounkov convex bodies and toric degenerations
- Schubert calculus

Family history

Mathematics Genealogy Project

Askold Georgievich Khovanskii

MathSciNet

Ph.D. Steklov Institute of Mathematics 1973

Dissertation: *Representability of Function in Quadratures*

Advisor: Vladimir Igorevich Arnold

Students:
Click here to see the students listed in chronological order.

Name	School	Year	Descendants
Borodich, Feodor	Lomonosov Moscow State University	1984	
Burda, Yuri	University of Toronto	2012	
Chulkov, Sergey	Stockholm University	2005	
Firsova, Tanya	University of Toronto	2010	
Gelfond, Olga	Lomonosov Moscow State University	1984	
Izadi, Farzali	University of Toronto	2001	6
Kaveh, Kiumars	University of Toronto	2002	3
Kiritchenko, Valentina	University of Toronto	2004	
Mazin, Mikhail	University of Toronto	2010	
Monin, Leonid	University of Toronto	2019	
Soprunov, Ivan	University of Toronto	2002	
Soprunova, Jenya	University of Toronto	2002	
Timorin, Vladlen	Steklov Institute of Mathematics	2003	1
Timorin, Vladlen	University of Toronto	2004	1

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http://dx.doi.org/10.4007/annals.2012.176.2.5

Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory

By Kiumars Kaveh and A. G. Khovanskii

To the memory of Vladimir Igorevich Arnold

Thank you for your attention!

Slides and references can be found on the webpage:

me.hse.ru/valkir